HUMBOLDT-UNIVERSITÄT ZU BERLIN

Resource Economics

Faculty of Life Sciences | Thaer-Institute

Introduction online course on Mathematics and Statistics

Preparatory Course for M.Sc. Integrated Natural Resource Management

1 Mathematical tools

Department for Agricultural Economics | Resource Economics Group

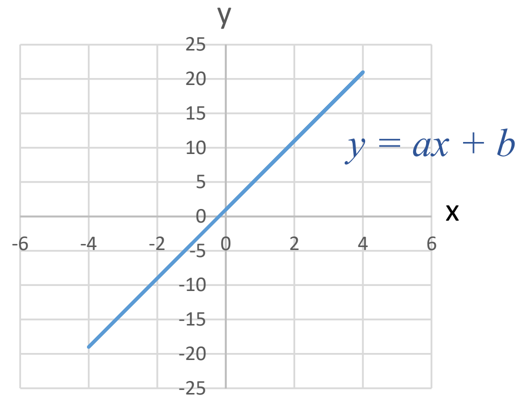
Syllabus

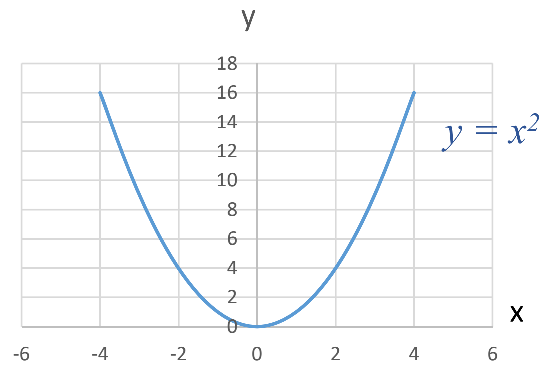
2 Mathematical tools

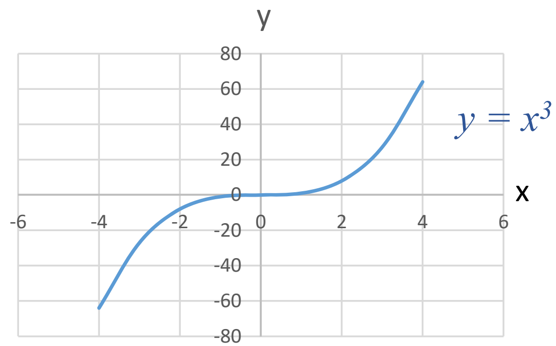
2.1 Function

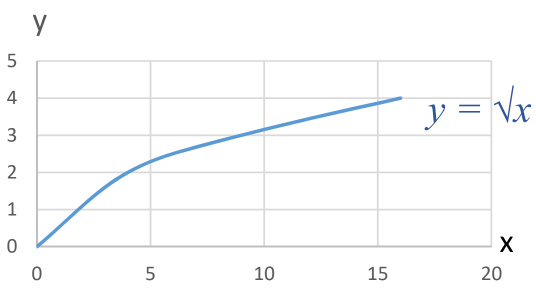
Compare: $y = x^2 – 2$ and $y^2 = x – 2$ [1].

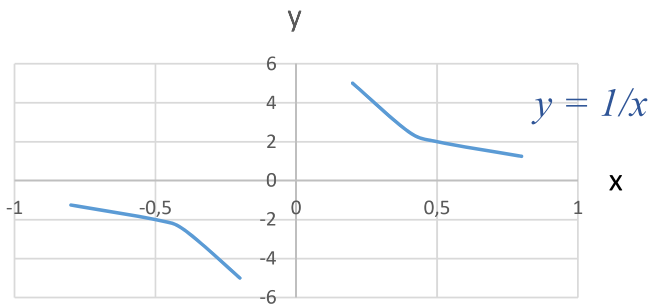
An equation will be a ***function*** if for every element of a set x there is exactly one element of a set y [1].

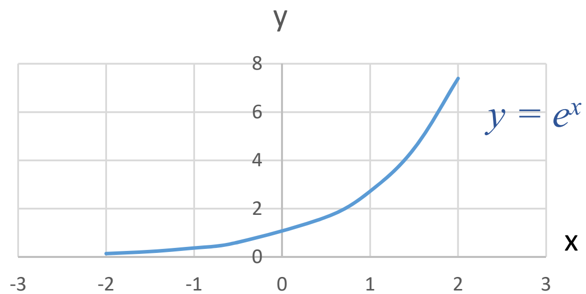












2.2 Limit of a function

$y = \frac{x^3 - 1}{x - 1}$ → $y(1) = \frac{1 - 1}{1 - 1}$ → $y(1) = \frac{0}{0}$ → This value is indeterminate [1].

x	$\frac{x^3 - 1}{x - 1}$
0,6	1,96
0,9	2,71
0,99	2,9701
0,999	2,997001
0,9999	2,9970001

→ $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$

x	$\frac{x^3 - 1}{x - 1}$
1,6	5,16
1,1	3,31
1,01	3,0301
1,001	3,003001
1,0001	3,00030001

→ $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$

The ***limit*** of a function *f(x)* is L as *x* approaches *p*, which means that *f(x)* gets closer and closer to *L* as *x* moves closer and closer to *p* [1].

$\lim_{x \to p} f(x) = L$

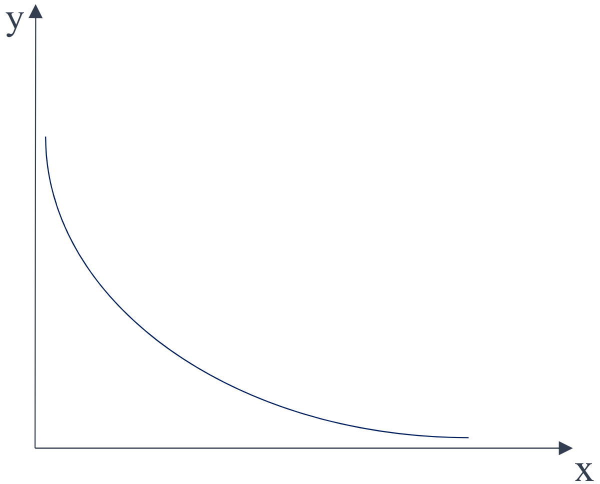
$f(x) \to L \quad x \to p$

$\lim_{x \to 6} \frac{x}{3} = 2$ → Limits can be still used though we know that 6/3 = 2 [2;3].

2.2 Limit of a function

- $f(x) = \frac{1}{x}$

x	y
1	1
2	0,5
4	0,25
10	0,1
100	0,01
1000	0,001
10000	0,0001
100000	0,00001



→ $\lim_{x\to\infty} f(x) = 0$ → $f(x) \to 0$ as $x \to \infty$

→ *f(x)* tends to a real limit *L* as *x* tends to infinity if, however small a distance we choose, *f(x)* gets closer than that distance to *L* and stays closer as *x* increases [3].

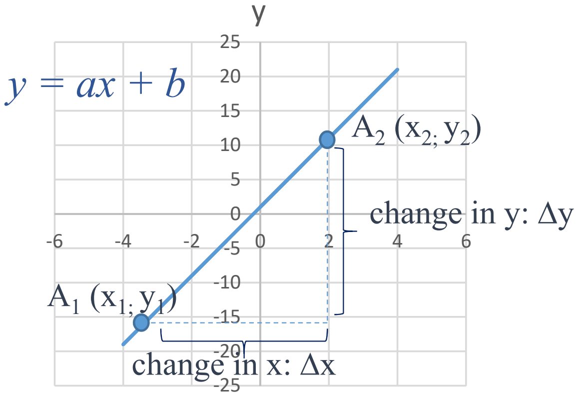
- $f(x) = x^2$ → $f(x) \to \infty$ as $x \to \infty$ → $\lim_{x\to\infty} f(x) = \infty$

→ *f(x)* tends to infinity as *x* tends to infinity if, however large a number we choose, *f(x)* gets larger than this number and stays larger, no matter how large *x* becomes [3].

- $f(x) = e^x$ → $f(x) \to 0$ as $x \to -\infty$

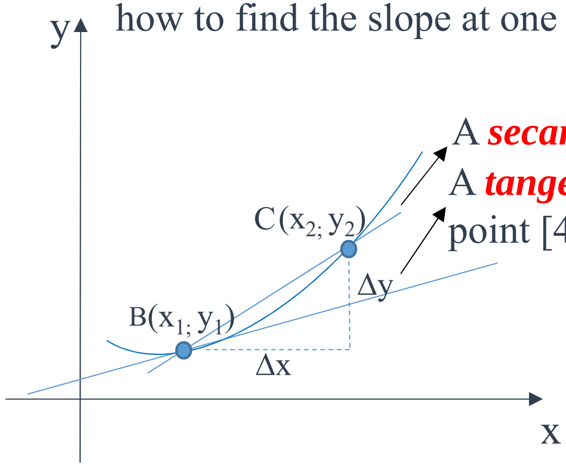
→ *f(x)* has a real limit *L* as *x* tends to minus infinity if, however small a distance we choose, *f(x)* gets closer than this distance to *L* and stays closer, no matter how large and negative *x* becomes [3].

2.3 Derivative of a function



$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$ → the slope of a straight line.

how to find the slope at one particular point? $\frac{\Delta y}{\Delta x} = \frac{0}{0}$ **?**

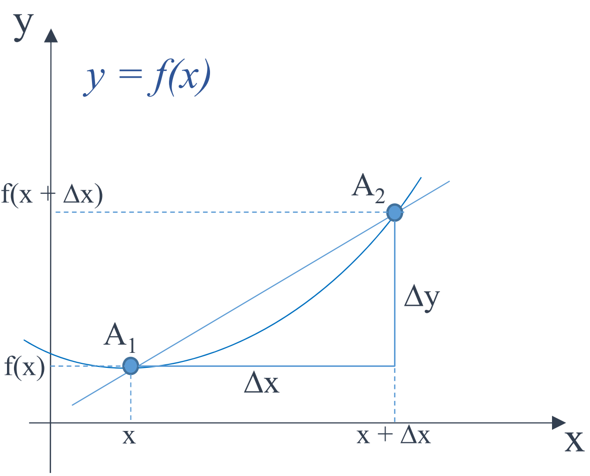


A ***secant line*** – the line that cuts a curve. The slope of the secant line: $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

A ***tangent line*** - the line that touches a curve at the point and is somehow 'parallel' to the graph at that point [4].

The slope of the tangent at A: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

2.3 Derivative of a function



$$\text{Slope of } A_1A_2 = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

→ The slope of the tangent line is the limit of the change in the function divided by the change in the independent variable as that change approaches 0 [4].

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$f(x) = x^2$

- $f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\,\Delta x + (\Delta x)^2$
- $\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \dfrac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \dfrac{2x\,\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$
- since $\Delta x \to 0$ → $f'(x) = 2x$

$$\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

2.3 Derivative of a function

- Derivative sum rule: $(a\,f(x) + bg(x))' = a\,f'(x) + bg'(x)$
- Derivative product rule: $(f(x) \cdot g(x))' = f'(x)\, g(x) + f(x)\, g'(x)$
- Derivative quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\, g(x) + f(x)\, g'(x)}{g^2\, x}$
- Derivative chain rule: $f(g(x))' = f'(g(x)) \cdot g'(x)$ [5]

Example: *find the derivative of function f given by $f(x) = (x^2 - 3)(x^3 - 2x + 7)$.*

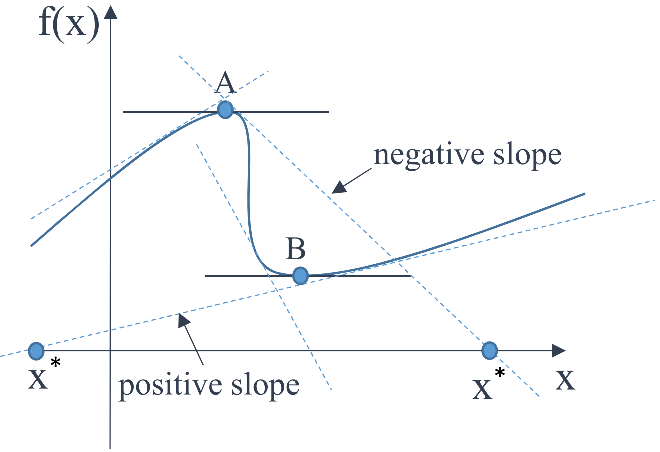
- *Function f is the product of two functions: $D = (x^2 - 3)$ and $E = x^3 - 2x + 7$ → the derivative product rule will be used.*
- *$f'(x) = 2x(x^3 - 2x + 7) + (x^2 - 3)(3x^2 - 2)$*
- *Expand: $f'(x) = 2x^4 - 4x^2 + 14x + 3x^4 - 2x^2 - 9x^2 + 6 = 5x^4 - 15x^2 + 14x + 6$*

Example: *find the derivative of function f given by $f(x) = \sqrt{x^5} + 6x - 3$.*

- *Function f is of the form of square root of D with $D = x^5 + 6x - 3$ → the derivative chain rule will be used.*
- *$f'(x) = D' / 2\sqrt{D}$*
- *$f'(x) = \frac{(5x + 6)}{2\sqrt{x^5 + 6x - 3}}$*

Function name	Function	Derivative
	f (x)	f '(x)
Constant	a	0
Linear	ax	a
Power	x^a	$a\,x^{a-1}$
Exponential	e^x	e^x
Exponential	a^x	$a^x \ln a$
Natural logarithm	ln(x)	1/x
Logarithm	$\log_b(x)$	1/x ln(b)
Sine	sin x	cos x
Cosine	cos x	-sin x

2.4 Maximizing and minimizing a function with one variable



- A ***necessary condition*** (first order condition): the condition is required, but the condition alone might not guarantee the result.
- A ***sufficient condition*** (second order condition): the presence of the condition is enough to guarantee a result.

→ A necessary condition for a point to be the max or the min of a function: $f'(x^*) = 0$.

→ A sufficient condition for a point to be the max of a function : $f'(x^*) < 0$.

→ A sufficient condition for a point to be the min of a function : $f'(x^*) > 0$ [6;7].

<u>Example:</u> Find the maximum and the minimum values of the function: $f(x) = x^3 - 12x - 2$ on the interval [-1,4].

- $f'(x) = 3x^2 - 12$
- A necessary condition: $f'(x^*) = 0$ → $3x^2 - 12 = 0$.
  $x^2 = 4$ → $x = 2$ or $x = -2$

- Since -2 is not included in the interval:
  $x^* = -1$ → $f'(x^*) = 9$
  $x^* = 2$ → $f'(x^*) = -18$
  $x^* = 4$ → $f'(x^*) = 14$
- Hence, the minimum is -18 when $x^* = 2$ and the maximum is 14 when $x^* = 4$.

2.5 Maximizing and minimizing a function with multiple variables

$f(x) = f(x_1, x_2, .., x_n)$

1) $\frac{df}{dx_1} = \frac{df(x_1, x_2, x_3, .., x_n)}{x_1} = 0$

2) $\frac{df}{dx_2} = 0$

…

n) $\frac{df}{dx_n} = 0$

Example: *A firm's profit from producing two different goods is determined by the function:* $p(x_1, x_2) = -2x_1^2 + 60x_1 - 3x_2^2 + 72x_2 + 100$. *At what quantity of each of these goods the optimal (maximum) situation will occur?*

- $\frac{dp}{dx_1} = -4x_1 + 60 = 0$
- $\frac{dp}{dx_2} = -6x_2 + 72 = 0$
- → *The optimal situation occurs when* $x_1 = 15$ *and* $x_2 = 12$ [8].

2.6 Constrained optimization

Decision problem: $\max_{x,y} f(x,y)$ s.t. g (x, y) ≥ 0 → **Lagrangian method:** L (x, y) = f (x, y) + λg (x, y)

(subject to) — s.t.; constraint — g (x, y) ≥ 0

The equation system: $\frac{dL}{dx} = 0, \quad \frac{dL}{dy} = 0, \quad g = 0.$

<u>Example:</u> *A consumer's preferences are given by the utility function* $U(x_1, x_2) = x_1^{1/2} x_2^{1/2}$, *where* x_1 *and* x_2 *are the consumed quantities of goods 1 and 2. Prices of these goods are consequently* $p_1 = 4$ *and* $p_2 = 2$. *Consumer's income is normally limited:* $\omega = 120$. *Maximize consumer's utility.*

- $\max_{(x_1, x_2)} U(x_1, x_2) \quad s.t.\ 4x_1 + 2x_2 = 120$
- $L(x_1, x_2) = x_1^{1/2} x_2^{1/2} + \lambda(120 - 4x_1 - 2x_2)$
- $L_{x_1}(x_1, x_2) = \frac{1}{2} x_1^{-1/2} x_2^{1/2} - 4\lambda = 0 \rightarrow \lambda = \frac{x_1^{-1/2} x_2^{1/2}}{8}$

  $L_{x_2}(x_1, x_2) = \frac{1}{2} x_1^{1/2} x_2^{-1/2} - 2\lambda = 0 \rightarrow \lambda = \frac{x_1^{1/2} x_2^{-1/2}}{4}$
- $\frac{x_1^{-1/2} x_2^{1/2}}{8} = \frac{x_1^{1/2} x_2^{-1/2}}{4} \rightarrow x_2 = 2x_1$
- $4x_1 + 2(2x_1) = 120 \rightarrow x_1 = 15 \rightarrow x_2 = 30 \rightarrow 4(15) + 2(30) = 120$

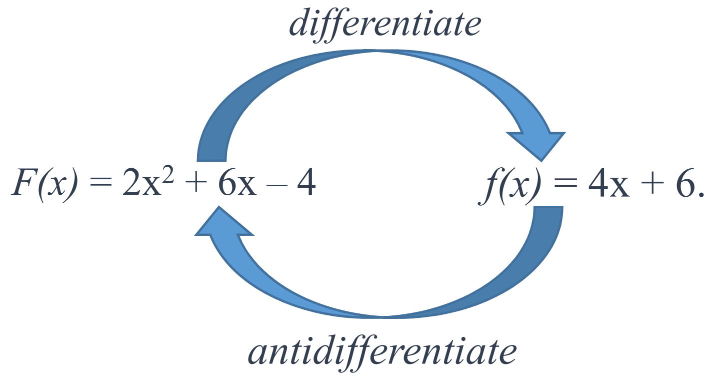
2.7 Introduction to integration

Integration allows solving two types of problems [11]:

- when the derivative of a function is known and there is a need to find the function → finding an indefinite integral;
- one might need to calculate areas, for instance, between a curve and x-axis or specific ordinates → definition of the definite integral.

Definition of an indefinite integral [11; 13]

- Finding an indefinite integral relates to the process of reverse differentiation.
- For instance, there is a given function *f(x)* and there is a need to find what function(s), *F(x)*, would have *f(x)* as its derivative.
- Let us consider the function $F(x) = 2x^2 + 6x - 4$. The derivative of this function will be then $f(x) = dF/dx$ → $f(x) = dF/dx = 4x + 6$.



- Which functions could possibly have 4x + 6 as a derivative? Certainly, the function $F(x) = 2x^2 + 6x - 4$ will be the answer.

→ If the derivative of F(x) is f(x), then an indefinite integral of f(x) with respect to x is F(x):

If $\frac{d}{dx}(F(x)) = f(x)$ then $\int f(x)dx = F(x)$.

→ If $\frac{d}{dx}(2x^2 + 6x - 4) = 4x + 6$ then $\int (4x + 6)dx = 2x^2 + 6x - 4$.

- However, such functions as $F(x) = 2x^2 + 6x$, $F(x) = 2x^2 + 6x + 12$, $F(x) = 2x^2 + 6x - 5$, etc. have the same derivative. The reason for this is the constant, which disappears during differentiation.
- So, if C is any constant → the derivative of $2x^2 + 6x + C$ is $4x + 6$, and consequently $2x^2 + 6x + C$ is an indefinite integral of $4x + 6$:

$$\int (4x + 6)dx = 2x^2 + 6x + C$$

2.7 Introduction to integration

- Key insights [13]:
  1) A function *F(x)* is an antiderivative of *f(x)* if d*F*/dx = *f(x)*;
  2) If *F(x)* is an antiderivative of *f(x)* then so too is *F(x)* + *C* for any constant *C*.

- Some general rules for calculating integrals (recall those of derivatives):
- Multiplication by constant: $\int (cf(x))dx = c \int f(x)dx$, for any constant c
- Sum rule: $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
- Power rule: $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$
- Difference rule: $\int (f(x) - g(x))\, dx = \int f(x)dx - g(x)dx$
- Integration by parts: $\int f(x)g(x)dx = f(x) \int g(x)dx - \int f'(x)\left(\int g(x)dx\right)dx$

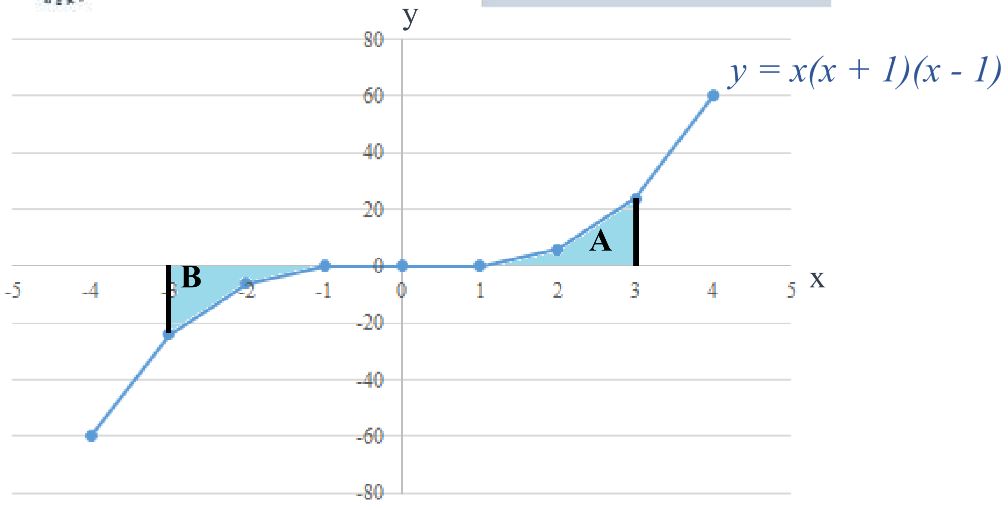
<u>Example:</u> Calculate the following integral: $\int \ln x/x^2$

- First we choose f(x) and g(x): $f(x) = \ln(x)$ and $g(x) = x^2$
- Differentiate f(x): $f'(x) = \ln(x) = 1/x$
- Integrate g(x): $\int 1/x^2\, dx = \int x^{-2}\, dx = -1/x$
- Substitute in the formula: $\ln x \frac{-1}{x} - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx = -\frac{\ln(x)+1}{x} + C$

Common antiderivatives [13]

$f(x)$	$F(x)$, an antiderivative of $f(x)$
k constant	$kx + C$
x	$\frac{x^2}{2} + C$
x^2	$\frac{x^3}{3} + C$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin mx$	$-\frac{1}{m}\cos mx + C$
$\cos mx$	$\frac{1}{m}\sin mx + C$
e^{mx}	$\frac{1}{m}e^{mx} + C$

2.7 Introduction to integration



Example: *calculate the areas of the segments between the curve of the function y = x(x + 1)(x - 1) and the x-axis.*

- $y = x(x+1)(x-1) = x^3 - x$
- $A = \int_1^3 y\,dx = \int_1^3 (x^3 - x)\,dx$

  $= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^3 = \left[\frac{3^4}{4} - \frac{3^2}{2}\right] - \left[\frac{1^4}{4} - \frac{1^2}{2}\right] = 16$
- $B = \int_{-1}^{-3} y\,dx = \int_{-1}^{-3} (x^3 - x)\,dx$

  $= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{-3} = 16$

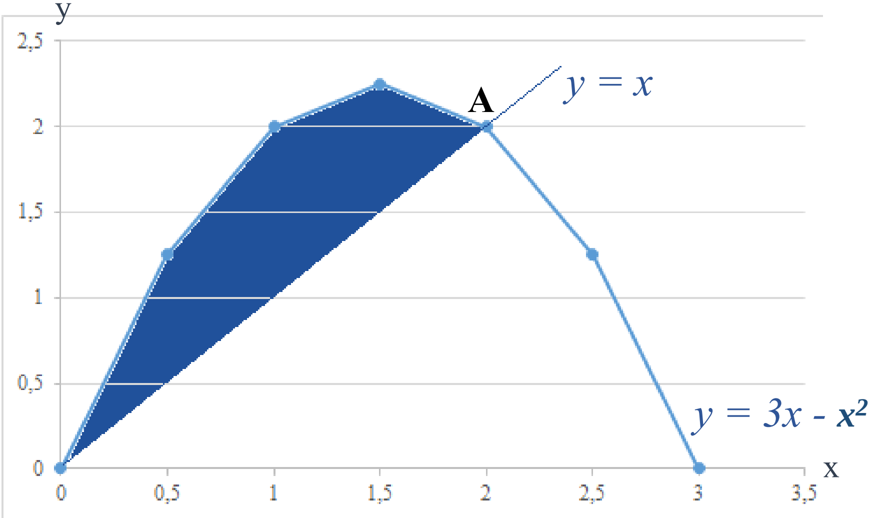
The total area: A + B = 16 + 16 = 32

oder

$A + B = \int_{-3}^{3} y\,dx = \int_{-3}^{3} (x^3 - x)\,dx = 16 - 16 = 0$

→ When calculating the area between a curve and the x-axis, separate calculations are required for each segment between the curve and the x-axis. It is therefore recommended to draw a sketch of the curve of a function to avoid such confusions and to exactly define how many separate calculations are needed [12].

2.7 Introduction to integration



Example: *calculate the area of the segment between the curve of a function $y = 3x - x^2$ and the line $y = x$.*

- *Find the points where the curves cross, so that the ordinates can be calculated → x = o and x = 2 (A).*
- *Find the area under the curve of the function:*

$$\int_0^2 y\,dx = \int_0^2 (3x - x^2)dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 = 3\frac{1}{3}$$

- *Find the area under the line:*

$$\int_0^2 y\,dx = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2$$

- *Subtract the area under the line $y = x$ from the area under the curve $y = 3x - x^2$:*

$$3\frac{1}{3} - 2 = 1\frac{1}{3}$$

→ The area between two curves is calculated by finding the area between one curve and the x-axis, and subtracting the area between the other curve and the x-axis. It is likewise useful to draw a sketch in this case in order to find the points where the curves cross for defining the limits for integration. It will also help in defining which areas should be subtracted [12].

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Exercise #1: Determine first derivatives of:

1) $f(x) = (\sqrt{x} + 2x)(4x^2 - 1)$

2) $f(x) = \dfrac{x^2 + 1}{5x - 3}$

3) $f(x) = \left(\dfrac{1}{x} - 3\right)\dfrac{x^2 + 3}{2x - 1}$

4) $f(x) = (x^3 + 4)^4$

5) $f(x) = (x^2 + 5)^{3/2}$ [9]

Exercise #2: Determine: $\max\limits_{[0,\infty]} f(x) = px - \dfrac{1}{2}x^2$

$\min\limits_{x\in[0,2]} f(x) = x^3 - 3x^2 + 3x - 1$

Exercise #3: A firm's production function is $f(x_{1,} x_2) = x_1^{2/3} x_2^{1/3}$, where x_1 represents capital units and x_2 working hours. The budget constraint of the firm is subjected to $100x_1 + 100_2 = 400000$. Use the Lagrangian method to define optimal solutions for the firm [8].

Answers to exercises

Exercise #1:

1) $f'(x) = (48x^{5/2} + 20x^2 - 4x^{1/2} - 1) / 2\sqrt{x}$
2) $f'(x) = (5x^2 - 6x - 5)(5x - 3)^2$
3) $f'(x) = (-6x^4 + 6x^3 + 17x^2 - 12x + 3)/(x^2)(2x - 1)^2$
4) $f'(x) = 12x^2 (x^3 + 4)^3$
5) $f'(x) = 3x (x^2 + 5)^{1/2}$

Exercise #2:

$x^* = p$, $f(x^*) = 1/2p^2$
$x^* = 1$, $f(x^*) = 0$

Exercise #3:

$x_1 = 2666$ and $x_2 = 1333$ and the maximum production will be then 2116 units.

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References:

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