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EXERCISES

in addition to the lecture in

mathematics

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Table of contents

1 Set-theoretic and arithmetic tools

1.1 Set theory

Graphic illustration of sets in the real number plane

Show the following sets as a graphic:

1) $S_1 = \{(x_1, x_2) \in R^2;\ x_1 + x_2 + 1 = 0\}$

Notice: R^2 designates the real number plane (a Cartesian coordinate system with x_1-axis and x_2-axis). The set of all points (x_1, x_2), which satisfy the equation $a_1x_1 + a_2x_2 = b$ (a_1, a_2, b real numbers), is a straight line in this number plane.

2) $S_2 = \{(x_1, x_2) \in R^2;\ x_2 = 3x_1 + 4\}$

3) $S_3 = \{(x_1, x_2) \in R^2;\ x_1 = 5\}$

4) $S_4 = \{(x_1, x_2) \in R^2;\ x_2 = -3\}$

5) $S_5 = \{(x_1, x_2) \in R^2;\ -x_1 + x_2 \leq 1\}$

Notice: The set of all points $(x_1, x_2) \in R^2$, which satisfy the inequality $a_1x_1 + a_2x_2 \leq (\geq) b$; a_1, a_2, b real numbers, is a half-plane H in the real number plane. To show this half-plane graphically, you have to draw the line $a_1x_1 + a_2x_2 = b$ first. This line divides the number plane into two half-planes. One of those is the requested half-plane H. To define H, you choose any point (x_{10}, x_{20}). This point may not be on the line $a_1x_1 + a_2x_2 = b$. After drawing the point (x_{10}, x_{20}) into the Cartesian coordinate system, it is to determine whether this point satisfies the inequality. That means: If $a_1x_{10} + a_2x_{20} < (>) b$, then the requested half plane is the half plane including the point (x_{10}, x_{20}). In contrast, if $a_1x_{10} + a_2x_{20} > (<) b$, then the point (x_{10}, x_{20}) is not in the requested half-plane H. That means the requested half-plane H is the other half plane which does not include the point (x_{10}, x_{20}).

6) $S_6 = \{(x_1, x_2) \in R^2;\ x_2 + 3x_1 \geq 6\}$

7) $S_7 = \{(x_1, x_2) \in R^2;\ -x_1 + x_2 \leq 1,\ 2x_2 - x_1 \geq 4\}$

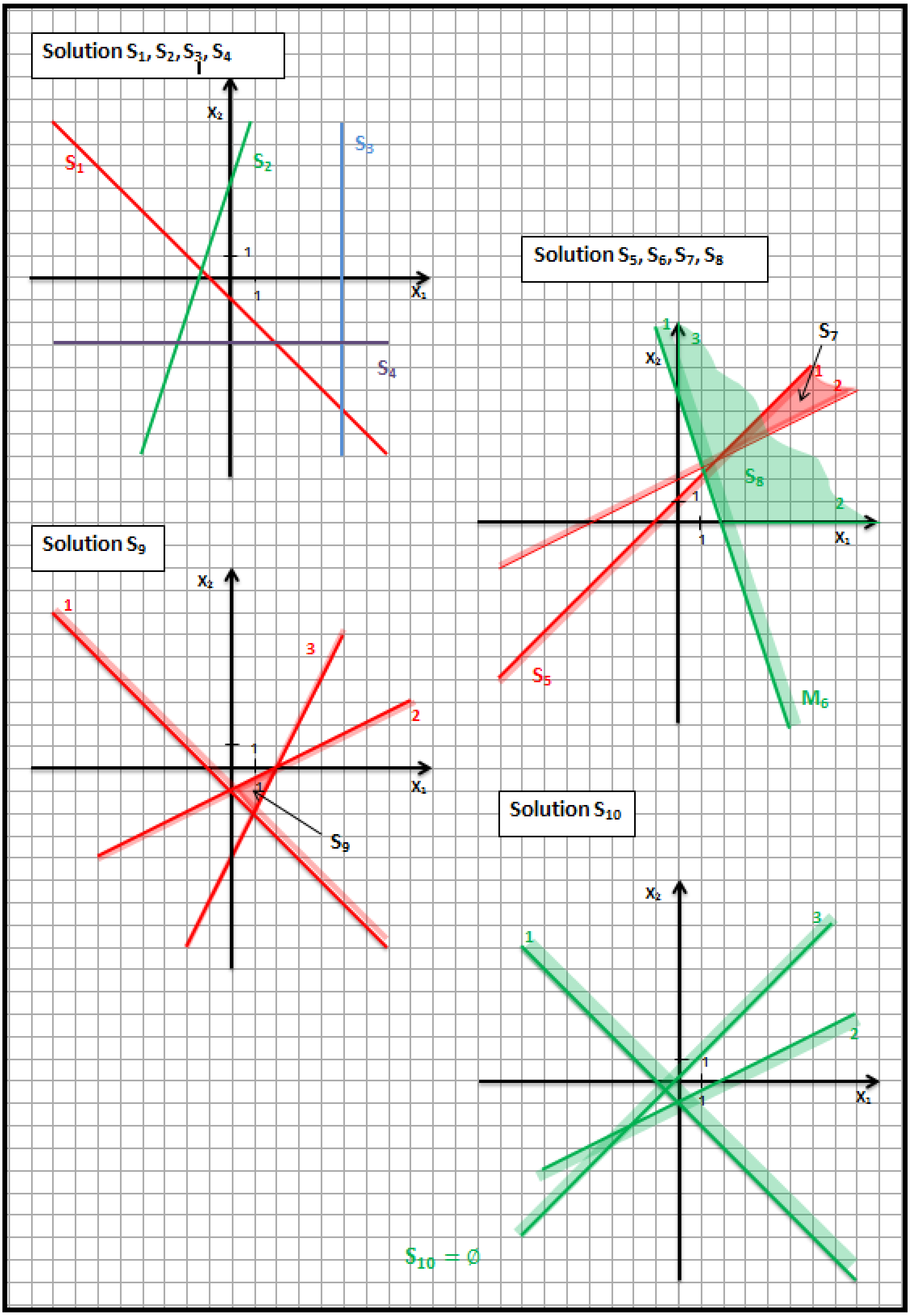
Notice: The set S_7 is equal to the set of all points (x_1, x_2) of the real number plane, which satisfy the inequality $-x_1 + x_2 \leq 1$ and the inequality $2x_2 - x_1 \geq 4$ simultaneously. The set S_7 is the intersection of both half-planes described by the inequalities.

8) $S_8 = \{(x_1, x_2) \in R^2;\ x_2 + 3x_1 \geq 6,\ x_1 \geq 0,\ x_2 \geq 0\}$

9) $S_9 = \{(x_1, x_2) \in R^2;\ x_1 + x_2 + 1 \geq 0,\ x_1 - 2x_2 - 2 \geq 0,\ 2x_1 - x_2 - 4 \leq 0\}$

10) $S_{10} = \{(x_1, x_2) \in R^2;\ x_1 + x_2 + 1 \geq 0,\ x_1 - 2x_2 - 2 \geq 0,\ x_2 - x_1 \geq 0\}$

Solutions



Solution S1, S2, S3, S4
x2
x1
1
S1
S2
S3
S4
Solution S5, S6, S7, S8
S5
M6
S7
S8
Solution S9
S9
Solution S10
S10 = Ø

1.2 The sigma notation

1) Write out the following sums and calculate.

a) $\sum_{i=1}^{7} i =$ $\qquad$ $\sum_{i=1}^{7} i + 2 =$ $\qquad$ $\sum_{i=1}^{7} (i+2) =$

b) $\sum_{k=1}^{3} 0{,}5 \cdot 2^{k} =$

c) $\sum_{i=1}^{4} 5 =$

d) $\sum_{i=0}^{3} \sum_{j=1}^{4} 2j(i+4) =$

e) $\sum_{n=-1}^{1} \sum_{k=2}^{4} (2+k)^{n} =$

2) Write the following sums by using the sigma notation.

a) $\frac{c_1}{d_1} + \frac{c_2}{d_2} + \frac{c_3}{d_3} + \ldots + \frac{c_{10}}{d_{10}} =$

b) $(a_1+b_1)^2+(a_2+b_2)^2+(a_3+b_3)^2+(a_4+b_4)^2+(a_5+b_5)^2=$

3) Suppose that i=1, 2, 3 designates three agricultural farms.
Suppose that j=1, 2, 3, 4 designates four consumers

The farms supply the consumers directly.

Be x_{ij} (dt) the amount of potatoes delivered by agricultural farm i to consumer j.

Be k_{ij} (€/dt) transport costs of one dt potatoes from agricultural farm i to consumer j.

a) Show x_{ij} in a table!

b) Indicate the formula describing the row sum R by using the sigma notation. What is the meaning of the first row sum?

c) Indicate the formula describing the column sum C by using the sigma notation. What is the meaning of the first column sum C_3?

d) Show the total amount of potatoes to be transported by using the sum over columns and by using the double sum sigma notation!

e) What is the meaning of $p_{ij}=k_{ij}x_{ij}$?
What is its unit?

f) Generate a similar table describing p_{ij}!

g) Set up a formula for sum P of all p_{ij}. Use the double sum sigma notation. What is the meaning of P?

h) What are the delivery costs of those potatoes delivered by agricultural farms 2 and 3 to consumer 1 and 2?

i) Calculate the requested values using this data :

x_{ij}:

Consumers

Farms	**1**	**2**	**3**	**4**
1	12	0	20	15
2	8	30	16	2
3	10	6	11	13

k_{ij}:

Consumers

Farms	**1**	**2**	**3**	**4**
1	20	62	57	50
2	48	25	30	80
3	78	75	40	45

4) x_i are measured values, i=1,..., n.

a) What is the meaning of $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$?

b) Prove the validity of $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^{\,2} - n(\bar{x})^2 = \sum_{i=1}^{n} x_i^{\,2} - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$ by using the equation $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

5) Analysis of variance is an important part of statistics and biometrics. An element of analysis of variance is to calculate several double sums.
In the following table you will find the yields (in kg) of five different varieties of tomatoes cultivated on 4 sections.
Calculate the double sums as stated below by using the tabular values.
The index i designates the number of variety (i=1,..., 5); n designates the number of section (n=1,...,4); x_{in} is the yield of variety i on section n.

x_{in}:

	n			
i	**1**	**2**	**3**	**4**
1	5,3	4,9	4,4	5,4
2	5,0	4,6	4,1	5,3
3	4,7	4,4	3,8	5,2
4	4,4	4,2	3,5	5,1
5	4,4	4,2	3,7	4,2

a) $\sum_i \sum_n x_{in}$

b) $\sum_i \sum_n {x_{in}}^2$

c) $(\sum_i \sum_n x_{in})^2$

d) $\sum_i (\sum_n x_{in})^2$

e) $\sum_n (\sum_i x_{in})^2$

Solutions

1a) 28; 30; 42 b) 7; c) 20; d) 440; e) $18\frac{37}{60}$

2a) $\sum_{i=1}^{10} \frac{c_i}{d_i}$ b) $\sum_{i=1}^{5} (a_i + b_i)^2$

3b) $R_1 = \sum_{j=1}^{4} x_{1j}$ c) $C_1 = \sum_{i=1}^{3} x_{i1}$ d) Total Set $C = \sum_{j=1}^{4} C_j = \sum_{j=1}^{4} \sum_{i=1}^{3} x_{ij}$

h) $p_{21}+p_{22}+p_{31}+p_{32}$

i) $R_1=47$, $C_1=30$, M=143, P=6159,

Transport costs: Farms 2 and 3 to Consumers 1 and 2: 2364 €

4b) Notice: It is best to rewrite the left side of the equation (binomial formula, assort and summarize the summands) until you get the middle part respectively the right part of the equation.

5a) 90,8 b) 418,20 c) 8244,64 d) 1656,7 e) 2079,02

2 Elements of mathematical analysis

2.1 Differential calculus

Find the derivatives of the following functions:

1) $f(x) = 3x^4 - 2x^3 + 4x - 2$

2) $f(\mathrm{x}) = \left(\mathrm{x}^2 - \frac{3}{\mathrm{x}}\right) \cdot \left(\mathrm{x}^2 + \frac{1}{\mathrm{x}}\right)$

3) $f(x) = (1 - ax)^4$

4) $f(x) = -3 \cdot \frac{1 - 2x}{(x^2 - 4)}$

5) $f(x) = \frac{\sin(x)^2}{\cos(x)}$

6) $f(x) = \frac{\sin(x^2)}{\cos(x)}$

7) $\sin(x) - x\cos(x)$

8) $f(x) = \frac{x}{a^x}$

9) $f(x) = e^x x^n$

10) $f(x) = \cos\left(e^{\frac{1-x}{1+x}}\right)$

Solutions

$f'(x) = 12x^3 - 6x^2 + 4$

$$f'(x) = \left(2x + \frac{3}{x^2}\right) \cdot \left(x^2 + \frac{1}{x}\right) + \left(x^2 - \frac{3}{x}\right) \cdot \left(2x - \frac{1}{x^2}\right)$$
$$= 4x^3 + \frac{6}{x^3} - 2$$

$f'(x) = -4(1 - ax)^3 \cdot a$

$$f'(x) = \frac{-6x^2 + 6x - 24}{(x^2 - 4)^2}$$

$$f'(x) = 2\sin(x) + \frac{\sin(x)^3}{\cos(x)^2}$$

$$f'(x) = \frac{2x\cos(x^2)}{\cos(x)} + \frac{\sin(x) \cdot \sin(x^2)}{\cos(x)^2}$$

$f'(x) = x\sin(x)$

$$f'(x) = \frac{1 - x\ln(a)}{a^x}$$

$f'(x) = e^x x^n + e^x x^{(n-1)} n$

$$f'(x) = -\sin\left(e^{\frac{1-x}{1+x}}\right) e^{\frac{1-x}{1+x}} \cdot \left[\frac{-2}{(1+x)^2}\right]$$

11) $f(x) = b \cdot e^{\sin(ax) - d}$

$f'(x) = b \cdot \cos(ax) a \cdot e^{\sin(ax) - d}$

12) $f(x) = x^n \ln(x)$

$f'(x) = n \cdot x^{n-1} \cdot \ln(x) + x^{n-1}$

13) $f(x) = x^2 \log_{10}(x)$

$f'(x) = 2x \frac{\ln(x)}{\ln(10)} + \frac{x}{\ln(10)} = 2x \log_{10}(x) + \frac{x}{\ln(10)}$

Notice: Given $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

14) $f(x) = \frac{1}{2}(\ln(x))^2$

$f'(x) = \frac{1}{2} \cdot 2\ln(x) \cdot \frac{1}{x} = \frac{\ln(x)}{x}$

2.2 Integral calculus

Find the set of antiderivatives of the following functions:

Solutions

1) $f(x) = ax^3 + bx + \frac{d}{x}$

$\int f(x)dx = \frac{1}{4}ax^4 + \frac{1}{2}bx^2 + d \cdot \ln(x) + c \; ; \; c \in R$

2) $f(x) = \frac{1}{x^n}$, $n \neq 1$

$\int f(x)dx = \frac{-1}{(n-1)} \cdot x^{(-n+1)} + c$

3) $f(x) = \frac{1}{x^n}$, $n = 1$

$\int f(x)dx = \ln x + c$

4) $f(x) = \sqrt{1 - \cos(x)^2}$

$\int f(x)dx = -\cos(x) + c$ (*Pythagoras*)

5) $f(x) = \frac{x^3 - 5x^2 + 12x - 12}{x - 2}$ $x \neq 2$

$\int f(x)dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 6x + c$ (*Polynomial division*)

6) $f(x) = \cot(x)$

Notice : $\cot(x) = \frac{\cos(x)}{\sin(x)}$

$\int f(x)dx = \ln|\sin(x)| + c$ $\left(f(x) = \frac{g'(x)}{g(x)} \right)$

7) $f(x) = \frac{2x - 5}{x^2 - 5x + 7}$

$\int f(x)dx = \ln|x^2 - 5x + 7| + c$ $\left(f(x) = \frac{g'(x)}{g(x)} \right)$

Calculate the definite integral.

8) $\int_0^3 2x + 3x^2 dx =$

9) $\int_0^{\frac{\pi}{2}} \cos(x) dx =$

10) $\int_1^2 \frac{1}{\sqrt{x}} dx =$

11) $\int_1^1 e^{2x} dx =$

12) What is the area between function $f(x)=x^2$ and function $g(x)=2x$?

36
1
$2 \cdot \sqrt{2} - 2$
0
$\int_0^2 g(x) - f(x) dx = 1{,}33\bar{3}$

2.3 Partial differentiation

1) Find the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1 \partial x_1}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2 \partial x_2}$ of each function.

a) $f(x_1, x_2) = x_1+x_2$

b) $f(x_1, x_2) = 3+5x_1+2x_2$

c) $f(x_1, x_2) = x_1 \cdot x_2$

d) $f(x_1, x_2) = (3+3x_1)(4-7x_2)$

e) $f(x_1, x_2) = -4x_1^2-x_2^2+3x_1x_2-2x_1-3x_2-4$

f) $f(x_1, x_2) = x_1^2(14x_1+6x_2)$

g) $f(x_1, x_2) = \frac{3x_1}{2x_2}+23$ $\qquad x_2 \neq 0$

h) $f(x_1, x_2) = x_1^3 \sin(x_2)+ x_1^2(1- x_2^3)^2$

i) $f(x_1, x_2) = x_1 \cdot e^{-(x_2+1)}$

j) $f(x_1, x_2) = \frac{4x_1x_2 + x_1^2}{x_1x_2^2}$ $\qquad x_1 \neq 0,\ x_2 \neq 0$

2) Find the extreme values of the functions $f(x_1, x_2)$ listed below. Proceed as described in the following:

1. Find $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$

2. Determine the stationary points $(x_{1E}, x_{2E}) \in R^2$, that means those points $(x_{1E}, x_{2E}) \in R^2$ satisfying the following condition:

$$\left.\begin{array}{l} \frac{\partial f}{\partial x_1}(x_{1E}, x_{2E}) = 0 \\ \frac{\partial f}{\partial x_2}(x_{1E}, x_{2E}) = 0 \end{array}\right\}$$

   The stationary points of a function $f(x_1, x_2)$ include the relative extreme values of a function. Now we have to examine for each stationary point if it is a relative extremum. If there is no stationary point, the function does not have any relative extremum.

3. Find (in case there are stationary points) the second-level partial derivatives .

$$\frac{\partial^2 f}{\partial x_1 \partial x_1}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2 \partial x_2}$$

4. For each stationary point (x_{1E}, x_{2E}), calculate the following:

$$\frac{\partial^2 f}{\partial x_1 \partial x_1}(x_{1E}, x_{2E}) = f_{x_1 x_1}(x_{1E}, x_{2E}) = a_{11}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_{1E}, x_{2E}) = f_{x_1 x_2}(x_{1E}, x_{2E}) = a_{12}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_2}(x_{1E}, x_{2E}) = f_{x_2 x_2}(x_{1E}, x_{2E}) = a_{22}$$

   If $D = a_{11} \cdot a_{22} - (a_{12})^2 > 0$, then we can establish a relative extremum in function $f(x_1, x_2)$ in point (x_{1E}, x_{2E}): It is a minimum if $a_{11} > 0$, it is a maximum if $a_{11} < 0$. If $D = a_{11} \cdot a_{22} - (a_{12})^2 < 0$, then we cannot establish a relative extremum of function $f(x_1, x_2)$ in point (x_{1E}, x_{2E}).

a) $f(x_1, x_2) = 3+2x_1+20x_2 - x_1^2 - 2x_2^2 - 3x_1x_2$

b) $f(x_1, x_2) = 9x_1^2+6x_1x_2+ 9x_2^2$

c) $f(x_1, x_2) = 3x_1^3+2x_1 - 4x_2^2+x_2 - x_1x_2$

d) $f(x_1, x_2) = 3x_1 \cdot x_2^2 - x_1^2 - 6x_2^2 - 8x_1$

e) $f(x_1, x_2) = e^{-(x_1-3)^2-(x_2+4)^2}$

Solutions

1)

	1a	**1b**	**1c**	**1d**	**1e**
$\frac{\partial f}{\partial x_1}$	1	5	x_2	$3(4-7x_2)$	$3x_2-8x_1-2$
$\frac{\partial f}{\partial x_2}$	1	2	x_1	$-7(3x_1+3)$	$-2x_2+3x_1-3$
$\frac{\partial^2 f}{\partial x_1 \partial x_1}$	0	0	0	0	-8
$\frac{\partial^2 f}{\partial x_1 \partial x_2}$	0	0	1	-21	3
$\frac{\partial^2 f}{\partial x_2 \partial x_2}$	0	0	0	0	-2

	1f	**1g**	**1h**
$\frac{\partial f}{\partial x_1}$	$2x_1(6x_2+14x_1)+14x_1^2$	$\frac{3}{2x_2}$	$3x_1^2\sin x_2+2x_1(1-x_2^3)^2$
$\frac{\partial f}{\partial x_2}$	$6x_1^2$	$-\frac{3x_1}{2x_2^2}$	$x_1^3\cos x_2-6x_1^2x_2^2(1-x_2^3)$
$\frac{\partial^2 f}{\partial x_1 \partial x_1}$	$2(6x_2+14x_1)+56x_1$	0	$6x_1\sin x_2+2(1-x_2^3)^2$
$\frac{\partial^2 f}{\partial x_1 \partial x_2}$	$12x_1$	$-\frac{3}{2x_2^2}$	$3x_1^2\cos x_2-12x_1x_2^2(1-x_2^3)$
$\frac{\partial^2 f}{\partial x_2 \partial x_2}$	0	$\frac{3x_1}{x_2^3}$	$-x_1^3\sin x_2+18x_1^2x_2^4-12x_1^2x_2(1-x_2^3)$

	1i	**1j**
$\frac{\partial f}{\partial x_1}$	e^{-x_2-1}	$\frac{1}{x_2^2}$
$\frac{\partial f}{\partial x_2}$	$-x_1e^{-x_2-1}$	$\frac{-2x_1-4x_2}{x_2^3}$
$\frac{\partial^2 f}{\partial x_1 \partial x_1}$	0	0
$\frac{\partial^2 f}{\partial x_1 \partial x_2}$	$-e^{-x_2-1}$	$-\frac{2}{x_2^3}$
$\frac{\partial^2 f}{\partial x_2 \partial x_2}$	$x_1e^{-x_2-1}$	$\frac{6x_1+8x_2}{x_2^4}$

2)

a) There is a stationary point at x_{1E} = 52, x_{2E} = -34, but we cannot establish an extremum at this point.

b) In point x_1=0, x_2=0 is a relative minimum.

c) There are no stationary points for function $f(x_1, x_2)$.

d) The function has three stationary points: (-4, 0); (2, 2); (2, -2).The function $f(x_1, x_2)$ has a relative maximum at (-4, 0). There are no extrema at the other stationary points.

e) $f(x_1, x_2)$ has a relative maxima at the stationary point (3, -4), which is the only stationary point.

3 Elements of linear algebra

3.1 Matrix calculus

1) Determine the type of the following matrixes.

$$\underline{A} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} 1 & 2 & 0 \\ 7 & 1/4 & 1 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix} \qquad \underline{d} = \begin{pmatrix} 1/2 \\ 1/4 \\ 1/8 \end{pmatrix} \qquad \underline{e} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

2) Transpose the following matrixes. What is the type of the transpose?

$$\underline{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix} \qquad \underline{d} = \begin{pmatrix} 8 \\ -1 \\ 0 \\ 4 \\ 3 \end{pmatrix}$$

$$\underline{c} = \begin{pmatrix} 3 & -1 & 10 \end{pmatrix}$$

3) Which matrixes are conformable for addition? Calculate the possible sums.

$$\underline{A} = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 0 & 2 \end{pmatrix}, \qquad \underline{a} = \begin{pmatrix} 2 & 5 & -6 \end{pmatrix}, \qquad \underline{B} = \begin{pmatrix} 17 & -10 & 13 \\ 0 & 7 & 2 \\ 23 & 5 & -8 \\ 33 & -11 & 15 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} -1 & 4 & 2 \\ 0 & 3 & 13 \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} -6 & 3 & 1{,}5 \end{pmatrix} \qquad \underline{F} = \begin{pmatrix} 0 & 1 & 8 \\ 0 & -4 & 2{,}5 \\ -41 & 17 & 23 \\ -3 & 28 & 0 \end{pmatrix}, \qquad \underline{D} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

4) Multiply the given matrix by the real number α.

a) $\alpha = 4$, $\quad \underline{a} = \begin{pmatrix} 3 & -1 & 10 \end{pmatrix}$

b) $\alpha = 3$, $\quad \underline{A} = \begin{pmatrix} 3 & 2 & 4 & 5 \\ 1 & 4 & 6 & 9 \\ -3 & 0 & 8 & 1 \end{pmatrix}$

c) $\alpha = 1/2$ $\quad \underline{b} = \begin{pmatrix} 4 & 7 & 3 & 5 \end{pmatrix}^T$

5) Write the matrix $\underline{A}$ as $\underline{A} = \alpha \cdot \underline{B}$, so that $\alpha \in R$ and matrix $\underline{B}$ consists only in integer elements.

$$\underline{A} = \begin{pmatrix} 1/3 & -1/12 & 1/6 \\ 1/4 & 1/8 & -1/24 \end{pmatrix}$$

6) We have one agricultural farm with four machines: The vector t describes the required machine time:

$$\underline{t} = \begin{pmatrix} 450 \\ 500 \\ 1000 \\ 740 \end{pmatrix},$$

the vector b describes the available machine time

$$\underline{b} = \begin{pmatrix} 500 \\ 600 \\ 1000 \\ 750 \end{pmatrix}$$

Determine the remaining machine time r

7) Four farms supply three customers monthly:

	Customer		
Farm	**1**	**2**	**3**
1	300	200	0
2	250	100	300
3	200	170	100
4	0	420	210

Determine the annual supply.

8) What is the type of matrix $\underline{X}$? Solve the following matrix equation.

$$3\begin{pmatrix} 1/3 & 0 \\ -4/6 & 1/2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^T = \underline{X}$$

9) What is the type of matrix $\underline{X}$? Solve the following matrix equation.

$$\frac{1}{4}\underline{X} - \begin{pmatrix} 3 & -6 & 0 \\ 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 & -4 \\ 0 & 19 & 3 \end{pmatrix}$$

10) The following vector equation shall be valid: Determine the real numbers a and b with:

$$a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

Matrix multiplication – Economic interdependencies

11) What is the type of matrix A and of matrix B? Is it possible to multiply as shown? Name the type of the product matrix and calculate if possible.

a) $\underline{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ $\underline{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\underline{A}\cdot\underline{B}$, $\underline{B}\cdot\underline{A}$

b) $\underline{A} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$ $\underline{B} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$ $\underline{A}\cdot\underline{B}$, $\underline{B}\cdot\underline{A}$

c) $\underline{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{A}\cdot\underline{b}$, $\underline{b}\cdot\underline{A}$

d) $\underline{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $\underline{a}\cdot\underline{b}$, $\underline{b}\cdot\underline{a}$

12) Calculate $\underline{A}\cdot\underline{A}^T$ if $\underline{A} = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$.

13) The vector $\underline{x}$ shall be $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. Determine the dimension of $\underline{x}$.

a) What is the dimension of $\underline{A}\cdot\underline{x}$, supposing that $\underline{A}$ is the matrix of exercise 12.

b) Calculate $\underline{A}\cdot\underline{x}$!

14) Give a matrix $\underline{A}$ with the type $(\underline{A}) = (2, 3)$ and a matrix $\underline{B}$ with the type $(\underline{B}) = (4, 2)$. Calculate the possible product matrix($\underline{A}\cdot\underline{B}$ or $\underline{B}\cdot\underline{A}$).

15) A company produces three intermediate products out of three resources in the first production stage. The three intermediate products are transformed to five final products in the second production stage.

a) Find the resource consumption matrix per unit of the final products.

b) Which amount of the resources must be provided if the company shall realise this vector $\underline{x}$ of the final products?

$$\underline{x} = \begin{pmatrix} 100 \\ 200 \\ 400 \\ 50 \\ 500 \end{pmatrix}$$

The material consumption of each production stage is shown below:

	I_1	I_2	I_3
R_1	2	1	4
R_2	0	5	3
R_3	3	2	0
R_4	4	1	2

	E_1	E_2	E_3	E_4	E_5
I_1	1	4	0	2	3
I_2	2	1	6	3	0
I_3	4	5	1	1	4

16) What is the type of $\underline{X}$? Determine the unknown matrix $\underline{X}$!

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \cdot \underline{X} = \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix}$$

17) What are possible product matrices? Calculate some.

$$\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \end{pmatrix} \qquad \underline{c}^T = \begin{pmatrix} 0 & 2 & 2 & -1 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} 5 & 4 & -2 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & -7 & 4 & 1 \end{pmatrix} \qquad \underline{D} = \begin{pmatrix} 4 & -2 \\ 2 & 0 \\ 0 & 2 \\ 3 & 1 \end{pmatrix} \qquad \underline{a} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix} \qquad \underline{C} = \begin{pmatrix} 3 & 1 & 5 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 4 & 1 & 1 & 0 \end{pmatrix}$$

$$\underline{H} = \begin{pmatrix} 4 & -2 \\ -2 & 0 \\ 0 & 2 \\ 3 & 1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \qquad \underline{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \underline{G} = \begin{pmatrix} 7 & 3 & 0 \\ 8 & 2 & 6 \\ -1 & -1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$

$$\underline{M} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & -1 \\ 3 & 2 & 0 \end{pmatrix} \qquad \underline{N} = \begin{pmatrix} 2 & 0 & -1 \\ -3 & 0 & 2 \\ -12 & 1 & 8 \end{pmatrix}$$

Solutions

5) $\underline{A} = \frac{1}{24}\begin{pmatrix} 8 & -2 & 4 \\ 6 & 3 & -1 \end{pmatrix}$

6) $\underline{r} = \underline{b} - \underline{t} = \begin{pmatrix} 50 \\ 100 \\ 0 \\ 10 \end{pmatrix}$

7)

	Customer		
Farm	**1**	**2**	**3**
1	3600	2400	0
2	3000	1200	3600
3	2400	2040	1200
4	0	5040	2520

8) Type ($\underline{X}$)=(2,2), $\underline{X}=\begin{pmatrix} 2 & 0 \\ -3 & 7/2 \end{pmatrix}$

9) Type ($\underline{X}$)=(2,3), $\underline{X}=\begin{pmatrix} 16 & 0 & -16 \\ 16 & 88 & 16 \end{pmatrix}$

10) $a = \frac{10}{3}$ $\quad b = \frac{-11}{3}$

11*a*) $\underline{A}\cdot\underline{B} = \begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix}$ $\quad \underline{B}\cdot\underline{A} = \begin{pmatrix} -1 & -1 \\ 5 & 3 \end{pmatrix}$

b) $\underline{A}\cdot\underline{B} = \begin{pmatrix} 9 & 3 \\ 10 & 3 \end{pmatrix}$ $\quad \underline{B}\cdot\underline{A} = \begin{pmatrix} 9 & 3 & 4 \\ 7 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$

c) $\underline{A}\cdot\underline{b} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$

d) $\underline{a}\cdot\underline{b} = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}$ $\quad \underline{b}\cdot\underline{a} = 13$

12) $\underline{A}\cdot\underline{A}^T = \begin{pmatrix} 18 & 21 \\ 21 & 27 \end{pmatrix}$

13) Type ($\underline{x}$) = (4,1) $\quad \underline{A}\cdot\underline{x} = \begin{pmatrix} 3x_1+2x_2+x_3+2x_4 \\ 4x_1+x_2+x_3+3x_4 \end{pmatrix}$ $\quad$ Type $(\underline{A}\cdot\underline{x}) = (2,1)$

15a) $\begin{pmatrix} 20 & 29 & 10 & 11 & 22 \\ 22 & 20 & 33 & 18 & 12 \\ 7 & 14 & 12 & 12 & 9 \\ 14 & 27 & 8 & 13 & 20 \end{pmatrix}$

b) $\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} 23350 \\ 26300 \\ 13400 \\ 20650 \end{pmatrix}$

16) Type ($\underline{X}$) = (2,2) $\quad \underline{X} = \begin{pmatrix} 2 & -23 \\ 0 & 8 \end{pmatrix}$

3.2 Linear independence – Basis of a vector space

1) Determine the vector x, which is a linear combination of the vectors $\underline{a}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\underline{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with the coefficients $\lambda_1 = 2,\ \lambda_2 = 3$. What is the dimension of vector x?

2) Determine the vector x, which is a linear combination of the vectors $\underline{a}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} -7 \\ 4 \\ 3 \end{pmatrix}$ with the coefficients $\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = -1$. What is the dimension of vector x?

3) The vector c shall be a linear combination of $\underline{a}_1$ and $\underline{a}_2$. Which λ_1 and λ_2 is requested?

a) $\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

b) $\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} a \\ b \end{pmatrix}$ $a, b \in R$

c) $\underline{a}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

d) $\underline{a}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4) Are the following sets of vectors linear independent or linear dependent? Examine it graphically and calculate it!

a) $\underline{a}_1 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

b) $\underline{a}_1 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

5) Are the following vectors linear independent?

a) $\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

6) Show that the following vectors are a basis in the three-dimensional vector space.

$\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Solutions

1) $\underline{x} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

2) $\underline{x} = \begin{pmatrix} 9 \\ -1 \\ -4 \end{pmatrix}$

3a) $\lambda_1 = 3,\ \lambda_2 = 2$ b) $\lambda_1 = a,\ \lambda_2 = b$ c) $\lambda_1 = 5,\ \lambda_2 = 8$ d) $\lambda_1 = 1,\ \lambda_2 = 1$

4a) $\underline{a}_1$ and $\underline{a}_2$ are linear independent b) $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ are linear dependent

5a) linear independent b) linear independent

6) $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ are linear independent.

3.3 Elementary transformation of a basis

Coordinates of a vector in a basis

1) Solve exercise 4), chapter 3.2 by using ETB.

2) Solve exercise 6), chapter 3.2 by using ETB.

3) Determine if the following sets of vectors are linear independent by using ETB

a) $\underline{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\underline{a}_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b) $\underline{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

c) $\underline{a}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

4) Show that $\underline{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ constitute basis in R^2. Give the coordinates of the following vectors in this basis.

Notice: The coordinates of a vector in a given basis $\underline{a}_1$, $\underline{a}_2$ are the coefficients λ_1 and λ_2 of the linear combination of this vector out of the basis vectors $\underline{a}_1$, $\underline{a}_2$.

a) $\underline{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ b) $\underline{a}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ c) $\underline{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ d) $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ e) $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

5) Show that the vectors $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ are a basis in R^3.

$\underline{a}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ $\underline{a}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ $\underline{a}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Give the coordinates of the following vectors in the basis $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$

a) $\underline{a}_1$ b) $\underline{a}_2$ c) $\underline{a}_3$

d) $\underline{b} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$ e) $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ f) $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ g) $\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Rank of a matrix

Determine the rank of the following matrices.

6) $\underline{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \end{pmatrix}$ 7) $\underline{A} = \begin{pmatrix} 4 & -2 & 6 \\ 6 & -3 & 9 \end{pmatrix}$ 8) $\underline{A} = \begin{pmatrix} 2 & 3 & 1 \\ -3 & -1 & -4 \\ 1 & 5 & 3 \end{pmatrix}$

9) $\underline{A} = \begin{pmatrix} -2 & -4 & 1 \\ 1 & 2 & -1/2 \\ 3 & 6 & -3/2 \end{pmatrix}$ 10) $\underline{A} = \begin{pmatrix} -2 & -4 & 0 \\ 1 & 5 & 1 \\ 3 & 3 & -1 \end{pmatrix}$ 11) $\underline{a} = \begin{pmatrix} -2 & -4 & 0 & 7 \end{pmatrix}$

12) $\underline{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 1 \end{pmatrix}$

13) GIve a matrix with the type (3,4), which

a) is of rank 1,

b) is of rank 2.

Solutions

3a) linear dependent   b) linear dependent   c) linear independent

4) $\underline{a}_1$, $\underline{a}_2$ are linear independent and are thus a basis in the two-dimensional vector space.

a) $\lambda_1 = 1,\ \lambda_2 = 0,$ b) $\lambda_1 = 0,\ \lambda_2 = 1$ c) $\lambda_1 = -5,\ \lambda_2 = 3$

d) $\lambda_1 = 3,\ \lambda_2 = -1$ e) $\lambda_1 = -2,\ \lambda_2 = 1$

5) $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ are linear independent and are thus a basis.

a) $\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 0$ b) $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 0$ c) $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 1$

d) $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 1$ e) $\lambda_1 = 1/3,\ \lambda_2 = -2/3,\ \lambda_3 = 1/3$

f) $\lambda_1 = -2/3,\ \lambda_2 = 4/3,\ \lambda_3 = 1/3$ g) $\lambda_1 = 4/3,\ \lambda_2 = -5/3,\ \lambda_3 = -2/3$

6) $r(\underline{A}) = 2$

7) $r(\underline{A}) = 1$

8) $r(\underline{A}) = 3$

9) $r(\underline{A}) = 1$

10) $r(\underline{A}) = 2$

11) $r(\underline{a}) = 1$

12) $r(\underline{a}) = 1$

3.4 Systems of linear equations

Solve the linear-equation systems below by using ETB. Proceed in the following way:

- Write the linear equation systems as vectors!
  (that means to determine the vectors $\underline{a}_1$, $\underline{a}_2$,..., $\underline{b}$)
- Create a starting table.
- Do the ETB by transforming much as many of the vectors $\underline{a}_i$ into the basis
- Determination of $r(\underline{A})$, $r(\underline{A}, \underline{b})$ and examination if the linear equation system is solvable
- If the linear equation system is solvable: Give the set of all solutions.
- If there are infinite solutions: Give the general solution and two specific solutions

1) $4x_1 + 3x_2 = 10$
$-x_1 + 5x_2 = 9$

2) $2x_1 + 4x_2 = 6$
$4x_1 + 8x_2 = 10$

3) $2x_1 + 4x_2 = 6$
$4x_1 + 8x_2 = 12$

4) $x_2 + x_3 = 3$
$x_1 + x_3 = 3$
$-x_1 + x_2 = 0$

5) $x_1 + 2x_3 = 2$
$x_1 + x_2 = 3$
$3x_1 + x_2 + x_3 = 4$

6) $x_1 + 3x_2 + x_3 = 6$
$-x_2 + 2x_3 = -4$
$x_1 + x_2 = 3$

7) $x_1 - 2x_2 + x_3 + x_4 = 1$
$x_1 - 2x_2 + x_3 - x_4 = -1$
$x_1 - 2x_2 + x_3 + 5x_4 = 5$

8) $2x_1 - x_2 + x_3 + x_4 = 1$
$x_1 + 2x_2 - x_3 + 4x_4 = 2$
$x_1 + 7x_2 - 4x_3 + 11x_4 = 1$

9) After two ETBs for the linear equation system

$$x_1 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 4 \\ -5 \\ 4 \\ 5 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \\ -4 \\ 1 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -2 \\ 8 \\ -13 \\ 12 \\ 9 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -4 \\ 4 \\ 2 \end{pmatrix}$$

$$\underline{a}_1 \qquad \underline{a}_2 \qquad \underline{a}_3 \qquad \underline{a}_4 \qquad \underline{b}$$

we got this table:

	$\underline{e}_1$	$\underline{e}_2$	$\underline{a}_3$	$\underline{a}_4$	$\underline{b}$
$\underline{a}_1$	1	0	2	-2	-1
$\underline{a}_2$	$-1/2$	$1/4$	-1	3	1
$\underline{e}_3$	$-3/2$	$5/4$	0	0	0
$\underline{e}_4$	2	-1	0	0	0
$\underline{e}_5$	$-1/2$	$-5/4$	0	0	0

Give the set of all solution of this linear equations system.

10) Give at least one linear equation system with 3 equations and 2 variables with the only solution $x_1 = 2$ and $x_2 = -1$.

Solutions

1) $x_1 = 1$, $x_2 = 2$

2) no solution

3) $x_1 = 3 - 2x_2$, $x_2 \in R$ arbitrary;
specific solutions: z.B. a) $x_2 = 0$, $x_1 = 3$, b) $x_2 = 4$, $x_1 = -5$

4) $x_1 = 3 - x_3$, $x_2 = 3 - x_3$, $x_3 \in R$ arbitrary

5) $x_1 = 0$, $x_2 = 3$, $x_3 = 1$

6) $x_1 = 1$, $x_2 = 2$, $x_3 = -1$

7) $x_1 = 2x_2 - x_3$, $x_4 = 1$, $x_2, x_3 \in R$ arbitrary

8) no solution

9) $x_1 = -1 - 2x_3 + 2x_4$ $x_2 = 1 + x_3 - 3x_4$ $x_3, x_4 \in R$ arbitrary

3.5 The inverse of a matrix

Determine the inverse of each of the following matrices. Verify your solution by using matrix multiplication.

1) $\underline{A} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$

2) $\underline{A} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \\ -3 & -8 & -4 \end{pmatrix}$

3) <u>Notice</u>: See exercise 1), chapter 3.4

$\underline{A} = \begin{pmatrix} 4 & 3 \\ -1 & 5 \end{pmatrix}$

4) <u>Notice</u>: See exercise 5), chapter 3.4

$\underline{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$

5) <u>Notice</u>: See exercise 6), chapter 3.4

$\underline{A} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

6) The matrix below is the inverse of the matrix <u>B</u>. Determine matrix <u>B</u>!

$\underline{B}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$

7) Solve exercise 16) chapter 3.1 by using matrix inversion!

Solutions

1) $\underline{A}^{-1} = \begin{pmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{pmatrix}$

2) $\underline{A}^{-1} = \begin{pmatrix} -4 & 4 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

3) $\underline{A}^{-1} = \begin{pmatrix} 5/23 & -3/23 \\ 1/23 & 4/23 \end{pmatrix}$

4) $\underline{A}^{-1} = \begin{pmatrix} -1/3 & -2/3 & 2/3 \\ 1/3 & 5/3 & -2/3 \\ 2/3 & 1/3 & -1/3 \end{pmatrix}$

5) $\underline{A}^{-1} = \begin{pmatrix} -2/5 & 1/5 & 7/5 \\ 2/5 & -1/5 & -2/5 \\ 1/5 & 2/5 & -1/5 \end{pmatrix}$

6) $\underline{B} = \left(\underline{B}^{-1}\right)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

3.6 Linear optimization

Solve the following linear optimization problem:

a) geometrically

b) arithmetically by using the simplex algorithm

1) $\max\left\{ x_1+2x_2 \,\middle|\, x_1+x_2\le 1,\ 2x_1-2x_2\le 1,\ x_1\ge 0,\ x_2\ge 0 \right\}$

2) $\max\left\{ 2x_1+x_2 \,\middle|\, -2x_1+x_2\le 1,\ x_1-2x_2\le 1,\ x_1\ge 0,\ x_2\ge 0 \right\}$

3) $\max\left\{ 5x_1+3x_2 \,\middle|\, 3x_1+5x_2\le 15,\ 5x_1+2x_2\le 10,\ x_1\ge 0,\ x_2\ge 0 \right\}$

4) $$\max\left\{ x_1+3x_2 \,\middle|\, \begin{array}{l} x_1-x_2\le 1 \\ 2x_1+x_2\le 2, \quad x_1\ge 0,\ x_2\ge 0 \\ x_1-x_2\ge 0, \end{array} \right\}$$

5) $\min\left\{ 2x_1-3x_2 \,\middle|\, -3x_1+x_2\le 1,\ 2x_1+x_2\le 6,\ x_1\ge 0,\ x_2\ge 0 \right\}$

6) $$\max\left\{ x_1+x_2 \,\middle|\, \begin{array}{l} x_1+3x_2\le 9 \\ 2x_1-x_2\le 2, \quad x_1\ge 0,\ x_2\ge 0 \\ x_1-2x_2\ge 2, \end{array} \right\}$$

Solutions

1) We consider the maximum of function f to be in point $x_1=0$, $x_2=1$ with the value 2.

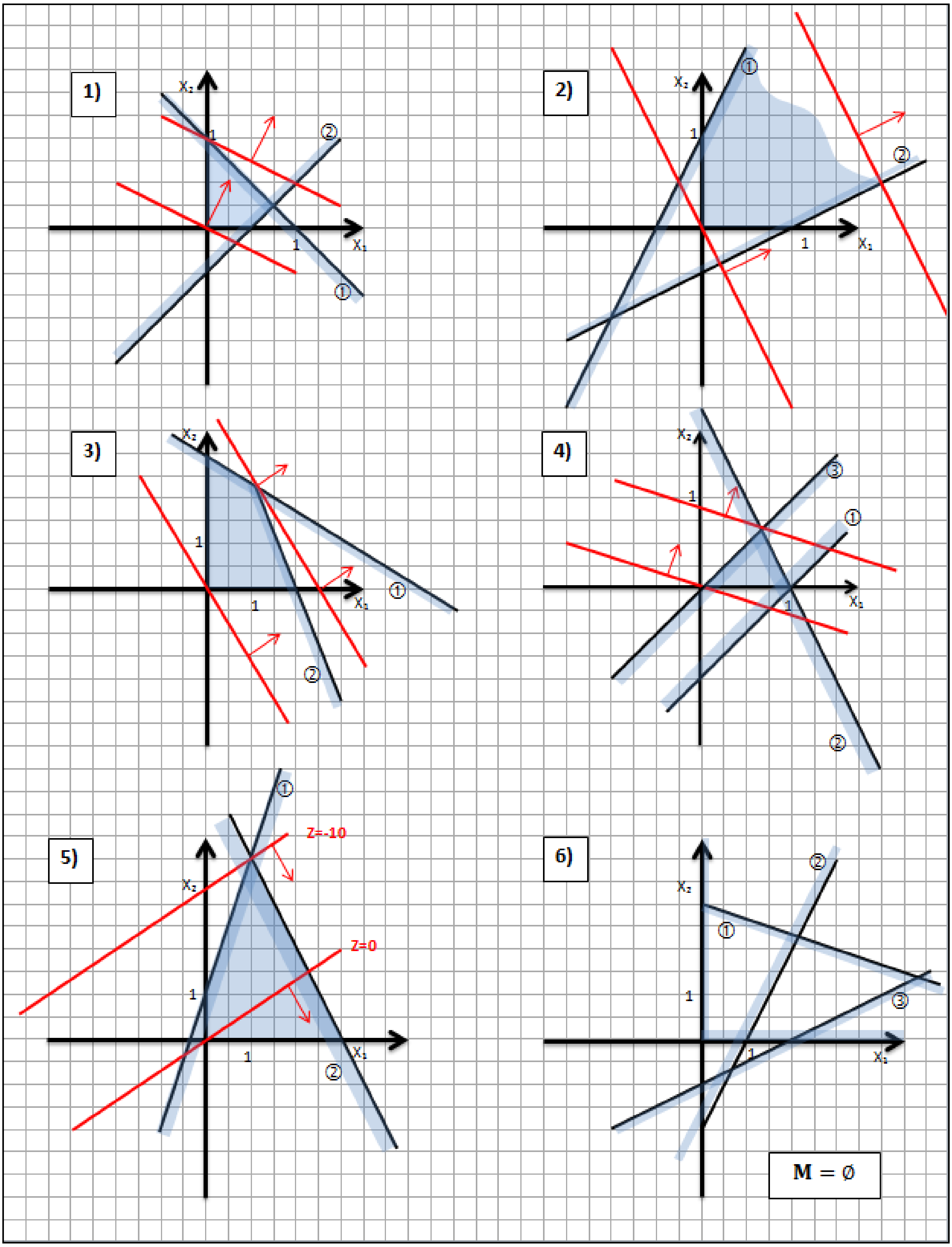
2) The set of feasible points is an unlimited convex polyhedron. There is no maximum solution. (Towards the objective function the set of feasible points is unlimited)

3) We consider the maximum of function f to be in point $x_1=\frac{20}{19}$, $x_2=\frac{45}{19}$ with the value $\frac{235}{19}$.

4) We consider the maximum of function f to be in point $x_1=\frac{2}{3}$, $x_2=\frac{2}{3}$ with the value $\frac{8}{3}$.

5) We consider the minimum of function $f(x_1, x_2)$ to be in point $x_1=1$, $x_2=4$ with the value -10.

6) The set of feasible points is empty, there is no solution.



1)
2)
3)
4)
5)
6)
Z=-10
Z=0
M = ∅