3.5 The inverse of a quadratic matrix

Definition: If for a quadratic matrix A there exists a quadratic matrix $\underline{A}^{-1}$ with $\underline{A}^{-1} \cdot \underline{A} = \underline{A} \cdot \underline{A}^{-1} = \underline{E}$, then $\underline{A}^{-1}$ is called the **inverse** of $\underline{A}$.

A regular matrix has exactly one inverse. If an inverse exists, then the matrix is regular.

$\underline{E}^{-1} = \underline{E}$

$(\underline{A}^{-1})^{-1} = \underline{A}$

Calculation of the inverse by using elementary transformations of a basis.
(There are other methods we will not deal with here: Gaussian elimination, Cramer's Rule):

	$\underline{a}_1$ ... $\underline{a}_n$
$\underline{e}_1$	
⋮	$\underline{A}$
$\underline{e}_n$	

After ETBs and → arranging the vectors

	$\underline{e}_1$ ... $\underline{e}_n$
$\underline{a}_1$	
⋮	$\underline{A}^{-1}$
$\underline{a}_n$	

Examples:

1) $\underline{A} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$

	$\underline{a}_1$	$\underline{a}_2$
$\underline{e}_1$	4	(2)
$\underline{e}_2$	5	3

	$\underline{a}_1$	$\underline{e}_1$
$\underline{a}_2$	2	1/2
$\underline{e}_2$	(-1)	-3/2

	$\underline{e}_2$	$\underline{e}_1$
$\underline{a}_2$	2	-5/2
$\underline{a}_1$	-1	3/2

ar-range →

	$\underline{e}_1$	$\underline{e}_2$
$\underline{a}_1$	3/2	-1
$\underline{a}_2$	-5/2	2

Thus $\underline{A}^{-1} = \begin{pmatrix} 3/2 & -1 \\ -5/2 & 2 \end{pmatrix}$ (Proof!)

2) $\underline{A} = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 0 & 1/2 \\ 0 & 1 & -1 \end{pmatrix}$

Compare with the coefficient matrix of the first equation system of chapter 3.4

Arrange $\Rightarrow$ $\underline{A}^{-1} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 3 & 2 & -2 \\ 3 & 2 & -3 \end{pmatrix}$ (Prove $\underline{A} \cdot \underline{A}^{-1}$ = ?)

Matrix equations with the inverse

- Linear equation system $\underline{A}\ \underline{x} = \underline{b}$

(1) Let's suppose $\underline{A}$ is regular, $\rho(\underline{A}) = n$, $\underline{x} \in R^n$:

$\underline{A}^{-1}$ exists and

$\underline{A}\ \underline{x} = \underline{b}$ is equivalent to

$\underbrace{\underline{A}^{-1}\ \underline{A}}_{\underline{E}}\ \underline{x} = \underline{A}^{-1}\ b$, that means $\underline{x} = \underline{A}^{-1}\ \underline{b}$ is the unique solution of the linear equation system

(2) Now let's suppose $\underline{A}$ is a (r, n)-matrix with $\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = r < n$:

we decompose $\underline{A}$ and $\underline{x}$ (by interchanging the corresponding columns and variables) in

$\underline{A} = (\underline{B}, \underline{N})$, so that $\underline{B}$ is a regular matrix, and

$\underline{x}$ in $\underline{x}_B$ and $\underline{x}_N$.

Out of $\underline{A}\ \underline{x} = \underline{b}$ respectively

$\underline{B}\ \underline{x}_B + \underline{N}\ \underline{x}_N = \underline{b}$ results via multiplication by $\underline{B}^{-1}$ from the left

$\underline{B}^{-1}\ \underline{B}\ \underline{x}_B + \underline{B}^{-1}\ \underline{N}\ \underline{x}_N = \underline{B}^{-1}\ \underline{b}$ respectively

$\underline{x}_B = \underline{B}^{-1}\ \underline{b} - \underline{B}^{-1}\ \underline{N}\ \underline{x}_N$.

The vector of the basic variables $\underline{x}_B$ is now represented by the vector of the non-basic variables $\underline{x}_N$, which is equivalent to the general solution of the linear equation system.

- Input / Output-analysis (Part 2)
  $(\underline{E} - \underline{A})\ \underline{x} = \underline{\mathbf{y}}$

(a) $\underline{x}$ is given, $\underline{y}$ is requested: matrix multiplication $\underline{y} = (\underline{E} - \underline{A})\ \underline{x}$

(b) $\underline{y}$ is given, $\underline{x}$ is requested: $\underline{x} = (\underline{E} - \underline{A})^{-1}\ \underline{y}$,
that means we need to calculate the inverse of $\underline{E} - \underline{A}$.