3.4 Linear equation systems

Definition: A system,

$$
\begin{array}{ccccccc}
a_{11} x_1 & + a_{12} x_2 & + & \dots & + a_{1n} x_n & = & b_1 \\
a_{21} x_1 & + a_{22} x_2 & + & \dots & + a_{2n} x_n & = & b_2 \\
\vdots & & & & \vdots & & \\
a_{m1} x_1 & + a_{m2} x_2 & + & \dots & + a_{mn} x_n & = & b_m
\end{array}
$$

with the constant parameters

a_{ij} for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$,

b_i for $i = 1, 2, ..., m$

and the variables

x_j for $j = 1, 2, ..., n$ is called

Linear equation system with m equations and n variables.

If all b_i for $i = 1, 2, ..., m$ are equal to 0,then the linear equation system is called **homogeneous**, otherwise it is called **inhomogeneous**.

Vector presentation:

$$\underline{a}_1 x_1 \quad + \underline{a}_2 x_2 \quad + \quad \ldots \quad + \underline{a}_n x_n \quad = \quad \underline{b}$$

$$\text{with} \quad \underline{a}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}, \text{für } i = 1, 2, \ldots, n \quad \text{und} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Matrix presentation:

$$\underline{A}\, \underline{x} = \underline{b}$$

$$\text{with} \quad \underline{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The term of a solution of a linear equation system:

Definition: a vector $x = \hat{x}$ of fixed values, which satisfies the condition $\underline{A}\, \underline{\hat{x}} = \underline{b}$ (which transfers it in an identity), is called a **solution** of the linear equation system $\underline{A}\, \underline{x} = \underline{b}$.

Example:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 5 \\ -3x_1 \qquad + \frac{1}{2} x_3 &= 10 \\ x_2 - x_3 &= 0 \end{aligned}$$

We solve the linear equation system by using elementary transformations of a basis:

	$\underline{a}_1$	$\underline{a}_2$ ↓	$\underline{a}_3$	$\underline{b}$
⇦ $\underline{e}_1$	2	(1)	–1	5
$\underline{e}_2$	–3	0	1/2	10
$\underline{e}_3$	0	1	–1	0

	$\underline{a}_1$	$\underline{e}_1$	$\underline{a}_3$ ↓	$\underline{b}$
$\underline{a}_2$	2	1	–1	5
⇦ $\underline{e}_2$	–3	0	(1/2)	10
$\underline{e}_3$	–2	–1	0	–5

	$\underline{a}_1$ ↓	$\underline{e}_1$	$\underline{e}_2$	$\underline{b}$
$\underline{a}_2$	–4	1	2	25
$\underline{a}_3$	–6	0	2	20
⇦ $\underline{e}_3$	(–2)	–1	0	–5

	$\underline{e}_3$	$\underline{e}_1$	$\underline{e}_2$	$\underline{b}$
$\underline{a}_2$	–2	3	2	35
$\underline{a}_3$	–3	3	2	35
$\underline{a}_1$	–1/2	1/2	0	5/2

$$\underline{\mathbf{b}} = \underline{\mathbf{a}}_1 \cdot \frac{5}{2} + \underline{\mathbf{a}}_2 \cdot 35 + \underline{\mathbf{a}}_3 \cdot 35$$

($x_1 = \frac{5}{2}$, $x_2 = 35$, $x_3 = 35$)

Solution: $$\underline{\mathbf{x}} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{5/2} \\ \mathbf{35} \\ \mathbf{35} \end{pmatrix}$$

(Proof!)

We had three equations and three variables in the example above. There was exactly one solution, which is not always the case.

Quadratic equation system: m=n	$\underline{A}$ singular ↓ it is possible ↙ ↘ $\nexists$(no solution) many solutions
Overdetermined equation system: m>n	it is possible mostly ↙ ↓ ↘ $\nexists$(no solution) $\exists!$ (exactly one solution) many solutions
Underdetermined equation system: m<n	it is possible mostly ↙ ↘ many solutions $\nexists$(no solution)

Two questions:

a) Solvability: Does the linear equation system have a solution?

b) Uniqueness: How many solutions does the linear equation system have?

Theorem: A linear equation system $\underline{A}\ \underline{x} = \underline{b}$ is solvable,if and only if $\rho(\underline{A}) = \rho(\underline{A},\underline{b})$

$\underline{A},\underline{b}$ is called extended coefficient matrix.

In the example above it is $\rho(\underline{A}) = 3 = \rho(\underline{A},\underline{b})$.

Theorem: Given the linear equation systems $\underline{A}\ \underline{x} = \underline{b}$ with n variables ($\underline{x} \in R^n$).
The linear equation system has exactly one solution if and only if $\rho(\underline{A}) = n$

If, in contrast, it holds $\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = r < n$, then $f = n - r$ variables are free to choose, viz. we have an infinite number of solutions.

f is the degree of freedom of the linear equation system

Examples:

1) task above: $\rho(\underline{A}) = 3 = n$

2) $3x_1 - x_2 + 2x_3 + x_4 = 5$

The linear equation system is composed of one equation with four variables.

$\rho(\underline{A}) = \rho(\underline{A},\underline{b}) = 1, \quad n = 4, \quad f = 3;$

$x_4 = 5 - 3x_1 + x_2 - 2x_3, \quad x_1, x_2, x_3 \in R$ arbitrary

3) $2x_1 + 2x_2 + 2x_3 = 1$

$x_1 - 4x_2 + 3x_3 - 2x_4 = -1$

The linear equation system is composed of two equations with four variables.

Solution of the linear equation system using elementary transformations of a basis:

	$\underline{a}_1$ ⇓	$\underline{a}_2$	$\underline{a}_3$	$\underline{a}_4$	$\underline{b}$
⇐ $\underline{e}_1$	(2)	2	2	0	1
$\underline{e}_2$	1	-4	3	-2	-1

	$\underline{e}_1$	$\underline{a}_2$	$\underline{a}_3$	$\underline{a}_4$ ⇓	$\underline{b}$
$\underline{a}_1$	1/2	1	1	0	1/2
⇐ $\underline{e}_2$	–1/2	-5	2	(-2)	-3/2

	$\underline{e}_1$	$\underline{a}_2$	$\underline{a}_3$	$\underline{e}_2$	$\underline{b}$
$\underline{a}_1$	1/2	1	1	0	1/2
$\underline{a}_4$	1/4	5/2	-1	-1/2	3/4

General solution:

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 5/2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, \qquad x_2, x_3 \in \mathbf{R} \quad \text{arbitrary}$$

Basis variable (BV) — Non-basic variable (NBV)

We get special solutions by fixing the arbitrary variables:

a) $x_2 = 1$, $x_3 = 1$ ⇒ $x_1 = -\frac{3}{2}$, $x_4 = -\frac{3}{4}$

thus $\underline{x} = \begin{pmatrix} -3/2 \\ 1 \\ 1 \\ -3/4 \end{pmatrix}$

b) $x_2 = 0$, $x_3 = 1$ ⇒ $x_1 = -\frac{1}{2}$, $x_4 = \frac{7}{4}$

thus $\underline{x}$ $\begin{pmatrix} -1/2 \\ 0 \\ 1 \\ 7/4 \end{pmatrix}$