3.3 Basis of a vector space, elementary transformation of a basis

Examples for n linear independent vectors in R^n:

$$\mathbf{R^2}: \quad \underline{\mathbf{a}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \underline{\mathbf{a}}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad ; \quad \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2\}$$

$\mathbf{R^n}$: $\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n$ standard basis

Definition: A set of n linear independent vectors in the n-dimensional vector space R^n is called **basis** of the vector space R^n

- The set of the following vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **and** $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ is not a basis.
- $\mathbf{B} = \{\underline{\mathbf{b}}_1, \ldots, \underline{\mathbf{b}}_n\}$ basis $\Rightarrow$ create all the linear combinations $\Rightarrow \mathbf{R^n}$
- If $\underline{\mathbf{c}} \in \mathbf{R^n} \Rightarrow$ we have a definite linear combination concerning the basis:

  $\underline{\mathbf{c}} = \mathbf{c}_1 \cdot \underline{\mathbf{b}}_1 + \ldots + \mathbf{c}_n \, \underline{\mathbf{b}}_n$

  $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are called coordinates concerning this basis.

Transition from one basis to another !?

Example:

$\mathbf{B}_1 = \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2\}$ → **$\underline{a}_2$ in the basis in exchange for $\underline{e}_1$** → $\mathbf{B}_2 = \{\underline{\mathbf{a}}_2, \underline{\mathbf{e}}_2\}$ → **$\underline{a}_1$ in the basis in exchange for $\underline{e}_2$** → $\mathbf{B}_3 = \{\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2\}$

Definition: We have one basis $\underline{\mathbf{b}}_1, \ldots, \underline{\mathbf{b}}_n$ out of R^n and another vector $\underline{\mathbf{a}} \in \mathbf{R}^n$.

The transition to a new basis is called elementary transformation of a basis, if it is possible to interchange the vector $\underline{\mathbf{a}}$ and the basis vector $\underline{\mathbf{b}}_i$, $\mathbf{i} \in \{1, \ldots, \mathbf{n}\}$, so that $\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \ldots, \underline{\mathbf{b}}_{i-1}, \underline{\mathbf{a}}, \underline{\mathbf{b}}_{i+1}, \ldots, \underline{\mathbf{b}}_n$ is a new basis of R^n.

Elementary transformation of a basis: Calculation tableau

		$\Downarrow$
	$\underline{a}_1$	$\underline{a}_2$
$\Leftarrow$ $\underline{e}_1$	2	1
$\underline{e}_2$	1	3

		$\Downarrow$
	$\underline{a}_1$	$\underline{e}_1$
$\underline{a}_2$	2	1
$\Leftarrow$ $\underline{e}_2$	-5	-3

I $\underline{a}_1 = 2\underline{e}_1 + \underline{e}_2$

II $\underline{a}_2 = \underline{e}_1 + 3\underline{e}_2$

II $\rightarrow$ $\underline{e}_1 = \underline{a}_2 - 3\underline{e}_2$

in I: $\underline{a}_1 = 2\underline{a}_2 - 6\underline{e}_2 + \underline{e}_2$

$= 2\underline{a}_2 - 5\underline{e}_2$

I $\underline{a}_1 = 2\underline{a}_2 - 5\underline{e}_2$

II $\underline{e}_1 = \underline{a}_2 - 3\underline{e}_2$

I $\rightarrow$ $5\underline{e}_2 = 2\underline{a}_2 - \underline{a}_1 \rightarrow \underline{e}_2 = \frac{2}{5}\underline{a}_2 - \frac{1}{5}\underline{a}_1$

in II: $\underline{e}_1 = \underline{a}_2 - 3\left(\frac{2}{5}\underline{a}_2 - \frac{1}{5}\underline{a}_1\right)$

$= -\frac{1}{5}\underline{a}_2 + \frac{3}{5}\underline{a}_1$

	$\underline{e}_2$	$\underline{e}_1$
$\underline{a}_2$	$\frac{2}{5}$	$-\frac{1}{5}$
$\underline{a}_1$	$-\frac{1}{5}$	$\frac{3}{5}$

$\xrightarrow{\text{arrange}}$

	$\underline{e}_1$	$\underline{e}_2$
$\underline{a}_1$	$\frac{3}{5}$	$-\frac{1}{5}$
$\underline{a}_2$	$-\frac{1}{5}$	$\frac{2}{5}$

(Algorithmic) rules R

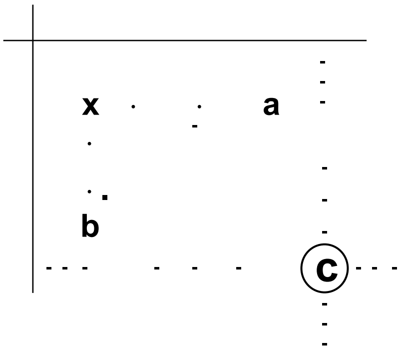
How do we get the elements (numbers, coordinates) of the new table out of the elements of the old tableau?

(1) **Central or pivot element** (c) shall be transformed to: $\frac{1}{c}$,

(2) The other elements of the **pivot row**
shall be multiplied by: $\frac{1}{c}$,

(3) The other elements of the **pivot column**
shall be multiplied by: $-\frac{1}{c}$,

(4) To get the remaining elements you have to use the **cross** rule:



x shall be transformed to : $x - \frac{a \cdot b}{c}$

Elementary transformation of a basis and linear dependence of vectors

Are the following vectors $\underline{\mathbf{a}}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\underline{\mathbf{a}}_2 = \begin{pmatrix} -10 \\ 2 \\ 3 \end{pmatrix}$, $\underline{\mathbf{a}}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

linear dependent or linear independent?

	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}_3$ ↓
$\underline{e}_1$	0	–10	–2
$\underline{e}_2$	1	2	0
← $\underline{e}_3$	–1	3	(1)

	$\underline{a}_1$ ↓	$\underline{a}_2$	$\underline{e}_3$
$\underline{e}_1$	–2	–4	2
← $\underline{e}_2$	(1)	2	0
$\underline{a}_3$	–1	3	1

	$\underline{e}_2$	$\underline{a}_2$	$\underline{e}_3$
$\underline{e}_1$	2	[0]	2
$\underline{a}_1$	1	2	0
$\underline{a}_3$	1	5	1

The central element shall differ from 0.

⇒ „Final tableau“

Evaluation of column $\underline{a}_2$:

$\underline{a}_2 = 2\underline{a}_1 + 5\underline{a}_3$

Theorem:

Let's suppose we have a basis in $\mathbf{R}^n$ and r additional vectors $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, ..., \underline{\mathbf{a}}_r \in \mathbf{R}^n$.

The vectors $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, ..., \underline{\mathbf{a}}_r$ are linear independent if it is possible to transfer them all together in the basis by using the elementary transformation of a basis.

The rank of a matrix

In general: The maximum number of linear independent rows of a matrix and the maximum number of linear independent columns of a matrix are identic.

Definition: The maximum number of linear independent columns (respectively rows) of matrix $\underline{A}$ is called **rank** ($\rho(\underline{A})$ also $r(\underline{A})$) of matrix $\underline{A}$.

Example:

$$\text{Let } \underline{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -5 & 1 \\ 1 & -1 & 1 \\ -2 & 1 & -3 \end{pmatrix}, \quad \rho(\underline{\mathbf{A}}) = ?$$

Determine the rank by using elementary transformation of a basis:

	$\underline{a}_1$ ↓	$\underline{a}_2$	$\underline{a}_3$
← $\underline{e}_1$	(1)	0	2
$\underline{e}_2$	3	–5	1
$\underline{e}_3$	1	–1	1
$\underline{e}_4$	–2	1	–3

	$\underline{e}_1$	$\underline{a}_2$ ↓	$\underline{a}_3$
$\underline{a}_1$	1	0	2
$\underline{e}_2$	–3	–5	–5
$\underline{e}_3$	–1	–1	–1
← $\underline{e}_4$	2	(1)	1

	$\underline{e}_1$	$\underline{e}_4$	$\underline{a}_3$
$\underline{a}_1$	1	0	2
$\underline{e}_2$	7	5	[0]
$\underline{e}_3$	1	1	[0]
$\underline{a}_2$	2	1	1

⇒ $\rho(\underline{A}) = 2$

Definition: A (n, n)-matrix $\underline{A}$ is called regular, if $\rho(\underline{A}) = n$.
If $\rho(\underline{A}) < n$, the matrix is called singular.

Another designation for “regular”: „matrix with full rank“.