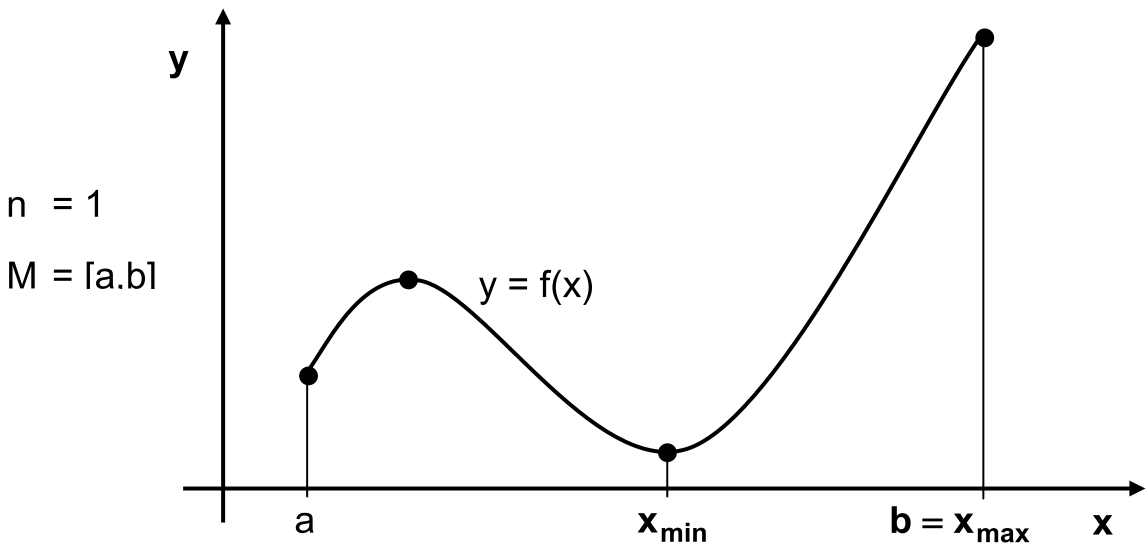
2.4 Generalizations, applications

a) Absolute extremum

Definition:

A function $\mathbf{f(\underline{x}): M \subset R^n \to R}$ has an absolute maximum at point $\mathbf{\underline{x}^0 \in M}$, if $\mathbf{f(\underline{x}^0) \geq f(\underline{x}) \quad \forall\ \underline{x} \in M}$.



In general the theorem of Weierstrass is valid:

If $\mathbf{f(\underline{x})}$ is a continous function in a restricted and closed set M, then an absolute maximum respectively minimum of f regarding M exists.

Methodology:

- consider $\mathbf{f(\underline{x})}$ in a compact (restricted and closed) set, e.g. a n-dimensionally cuboid,
- determine all local extremes,
- compare those to the values of f at the margin of set M.

b) Taylor's theorem

Describing the value of a function in the surrounding of a known point by using the (partial) derivative.

$\mathbf{n = 1: x_0 \in R, f(x_0)}$ and the derivatives $\mathbf{f^{(k)}(x_0)}$

until the order m are given, considering

$$\mathbf{x_0 + h \in R,}$$

$$\mathbf{f(x_0 + h) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k\,!} \cdot h^k + R_{m+1}}$$

$$\mathbf{n > 1:\ \underline{x}^0 = (x_1^0,\ x_2^0,\ \ldots,\ x_n^0) \in R^n,}$$

$$\mathbf{f(\underline{x}) : R^n \to R, \qquad \underline{h} = (h_1, h_2, \ldots, h_n) \in R^n}$$

The first and second order „partial“ derivatives of f in point $\mathbf{\underline{x}^0}$ are given,

$$\mathbf{f(\underline{x}^0 + \underline{h}) = f(x_1^0 + h_1,\ x_2^0 + h_2,\ \ldots,\ x_n^0 + h_n)}$$

$$\mathbf{= f(\underline{x}^0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}^0) \cdot h_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\ \partial x_j} (\underline{x}^0) \cdot h_i \cdot h_j + R}$$

c) Relative constrained extremum

$\mathbf{f}(\underline{x}): R^n \to R, \quad g_i(\underline{x}): R^n \to R, \; i = 1, \ldots, m$,

let the partial derivatives be continuous,

$$f(\underline{x}) \to \min\ (\max)$$

Under the conditions:

$$\begin{aligned} g_1(\underline{x}) &= 0 \\ g_2(\underline{x}) &= 0 \\ &\vdots \\ g_m(\underline{x}) &= 0 \end{aligned}$$

We consider for the **Lagrange-Funktion**

$$L(\underline{x}, \underline{\lambda}) := f(\underline{x}) + \lambda_1 \, g_1(\underline{x}) + \ldots + \lambda_m \, g_m(\underline{x})$$

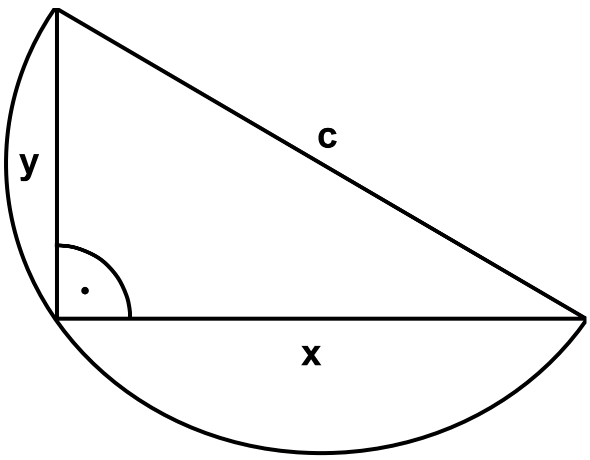
the necessary conditions for a relative extremum:

$$\frac{\partial L}{\partial x_i} = 0 \quad , \text{ für } i = 1, \ldots, n,$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \quad , \text{ für } j = 1, \ldots, m$$

and solve this system of equations.

Example:



Given the hypotenuse c, for which x,y the surface of the triangle is at its maximum?

$$f(x,y) = \frac{x \cdot y}{2} \rightarrow \max$$

Under the conditions $x^2 + y^2 = c^2$

$$L(x, y, \lambda) = \frac{x \cdot y}{2} + \lambda \cdot (x^2 + y^2 - c^2)$$

$$\frac{\partial L}{\partial x} = \frac{y}{2} + 2\lambda x = 0 \qquad (1)$$

$$\frac{\partial L}{\partial y} = \frac{x}{2} + 2\lambda y = 0 \qquad (2)$$

(1) $\rightarrow \quad \lambda = -\frac{y}{4x}$

add in (2): $\frac{x}{2} - \frac{y^2}{2x} = 0 \quad \rightarrow \quad x^2 = y^2$

with $x^2 + y^2 = c^2 \quad (\frac{\partial L}{\partial \lambda} = 0)$ follows $x^2 = y^2 = \frac{c^2}{2}$,

this means x = y (isosceles triangle).

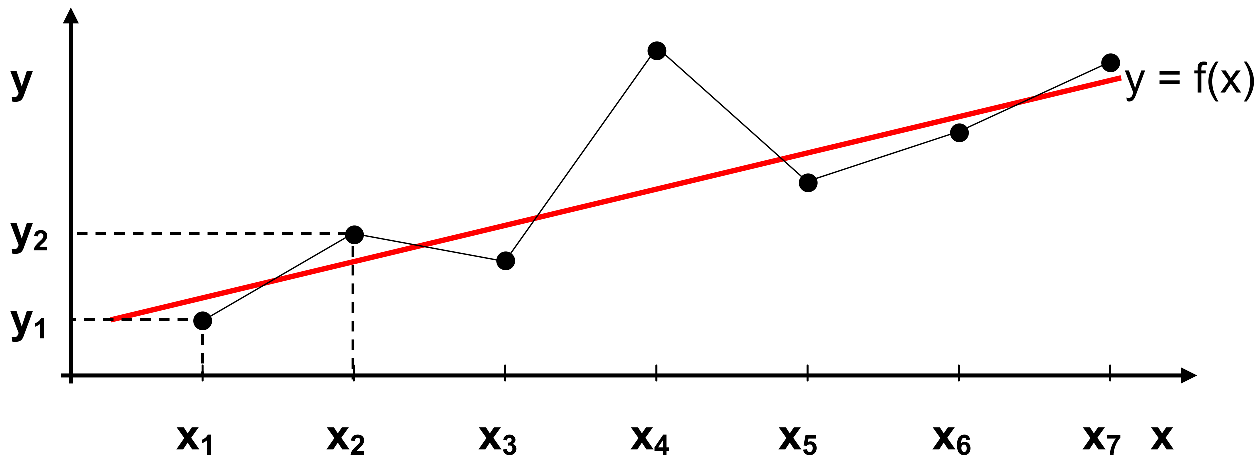
d) Analysis of trend and regression

- Investigation of behaviour respectively of change of certain data or value– economical, biological i.a.

  e.g. Gross national product, saving deposit, annual milk consumption of the population etc.

- List of time series**:**

year x	1994	1995	1996	1997	1998	1999	2000
data y	y_1	y_2	y_3	y_4	y_5	y_6	y_7



Requested: functional „dependency“ of y regarding x

- Different functional models are possible:
  - assumption of a linear connection; we are looking for a linear function $y = ax + b$, so that the given points „are in the surrounding of the function” this means we are looking for the parameters a and b; such a function is called linear trend function (it is also called adjustment of data, fitting respectively equalization calculus)

-Assumption: the function we are looking for is a polynomial

$$\mathbf{y} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{n}} \mathbf{a_i}\, \mathbf{x^i},$$

this means we are looking for n, a_0, a_1, ..., a_n;

n = 2 quadratical adjustment

n = 3 cubical adjustment.

- we are looking for a good adjustment using an exponential function (e.g. regarding heavy increase) $y = a \cdot e^{bx}$, this means we are looking for the corresponding parameters a and b.

- So far we considered time steps x_1, x_2, x_3, ..., with similar gaps (e.g. annual measures).⇒ **Trend**

  Now we consider two features (random parameters) x and y.

  ⇒ **Regression**

e.g. (1) x – weight of an animal in a population

y – size of the animal.

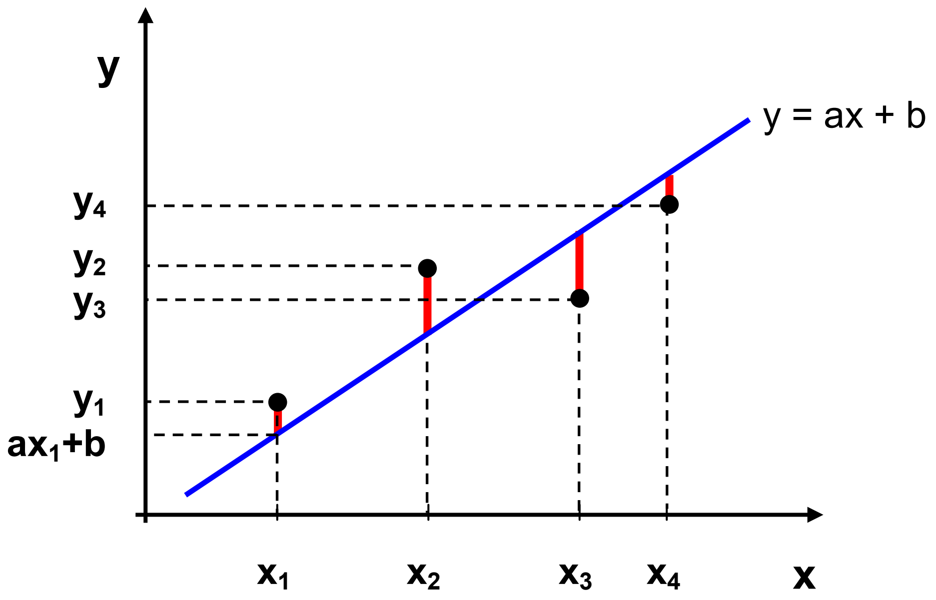
We are talking about a random sample x_i, y_i for i = 1, 2,...,k

(2) x – precipitation amount

y – harvest of an agricultural good(ZR).

Question: Does a connection exist between x and y?

Of which type is this connection, maybe linear?



$$\varepsilon_1 = y_1 - (ax_1 + b) \rightarrow \varepsilon_1^2 = (y_1 - ax_1 - b)^2$$
$$\varepsilon_2 = y_2 - (ax_2 + b) \rightarrow \varepsilon_2^2 = (y_2 - ax_2 - b)^2$$
$$\varepsilon_3 = ax_3 + b - y_3 \rightarrow \varepsilon_3^2 = (y_3 - ax_3 - b)^2$$
$$\varepsilon_4 = ax_4 + b - y_4 \rightarrow \varepsilon_4^2 = (y_4 - ax_4 - b)^2$$

$$\text{Sum:} \quad \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

We do now have a function with two variables (a is the ascent and b is the intersection point with the y-axis of the requested function y = ax + b), which shall be minimized:

$$F = F(a,b) = \sum_{i=1}^{n} (y_1 - ax_i - b)^2 \rightarrow \min.$$

The method of least squares

$$\mathbf{F(a,b)} = \sum_{i=1}^{n} (y_i - a \cdot x_i - b)^2 \rightarrow \mathbf{Min}$$

at a given random sample x_i , y_i for i = 1,...,n

$$\frac{\partial F}{\partial b} = \sum_{i=1}^{n} 2(y_i - ax_i - b) \cdot (-1) = -2\sum_{i=1}^{n}(y_i - ax_i - b)$$

$$-2\sum_{i=1}^{n}(y_i - ax_i - b) = 0 \qquad \Big| : (-2)$$

$$\sum_{i=1}^{n}(y_i - ax_i - b) = 0$$

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} ax_i - \sum_{i=1}^{n} b = 0 \qquad \Big| \sum_{i=1}^{n} b = n \cdot b$$

$$\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i = nb \qquad \Big| : n$$

$$b = \overline{y} - a\overline{x} \qquad \Big| \overline{x} = \tfrac{1}{n}\sum_{i=1}^{n} x_i \qquad \overline{y} = \tfrac{1}{n}\sum_{i=1}^{n} y_i \qquad \text{average values}$$

The function y=ax+b intersects the point

$$(\overline{x}, \overline{y})$$

$$F(a,b)=\sum_{i=1}^{n}(y_i-a\cdot x_i-b)^2 \to \mathrm{Min}$$

$$\frac{\partial F}{\partial a}=\sum_{i=1}^{n}2(y_i-ax_i-b)\cdot(-x_i)=-2\sum_{i=1}^{n}(y_i-ax_i-b)x_i$$

$$-2\sum_{i=1}^{n}(y_i-ax_i-b)x_i=0 \qquad \Big|:(-2)$$

$$\sum_{i=1}^{n}(y_i-ax_i-b)x_i=0$$

$$\sum_{i=1}^{n}y_ix_i-\sum_{i=1}^{n}ax_i^{\,2}-\sum_{i=1}^{n}bx_i=\sum_{i=1}^{n}y_ix_i-a\sum_{i=1}^{n}x_i^{\,2}-b\sum_{i=1}^{n}x_i=0$$

$$\sum_{i=1}^{n}y_ix_i-a\sum_{i=1}^{n}x_i^{\,2}-(\overline{y}-a\overline{x})\sum_{i=1}^{n}x_i=0$$

$$\sum_{i=1}^{n}y_ix_i-\overline{y}\sum_{i=1}^{n}x_i=a\Big(\sum_{i=1}^{n}x_i^{\,2}-\overline{x}\sum_{i=1}^{n}x_i\Big) \qquad \Big|:\Big(\sum_{i=1}^{n}x_i^{\,2}-\overline{x}\sum_{i=1}^{n}x_i\Big)$$

$$a=\frac{\sum_{i=1}^{n}y_ix_i-\overline{y}\sum_{i=1}^{n}x_i}{\Big(\sum_{i=1}^{n}x_i^{\,2}-\overline{x}\sum_{i=1}^{n}x_i\Big)} \qquad b=\overline{y}-\frac{\sum_{i=1}^{n}y_ix_i-\overline{y}\sum_{i=1}^{n}x_i}{\Big(\sum_{i=1}^{n}x_i^{\,2}-\overline{x}\sum_{i=1}^{n}x_i\Big)}\cdot\overline{x}$$

$$\left(a=\frac{\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\sum_{i=1}^{n}(x_i-\overline{x})^2} \qquad , \qquad y=\overline{y}+a(x-\overline{x})\right)$$

Checking the sufficient condition

$$\frac{\partial^2 F}{\partial a \partial a} = 2\sum_{i=1}^{n} x_i^2 > 0 \qquad \frac{\partial^2 F}{\partial a \partial b} = \frac{\partial^2 F}{\partial b \partial a} = 2\sum_{i=1}^{n} x_i \qquad \frac{\partial^2 F}{\partial b \partial b} = 2n$$

$$D = 2\sum_{i=1}^{n} x_i^2 \cdot 2n - 4\left(\sum_{i=1}^{n} x_i\right)^2 = 4n \cdot \left(\sum_{i=1}^{n} x_i^2 - \tfrac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i\right) \right)$$

$$= 4n \cdot \left(\sum_{i=1}^{n} x_i^2 - \overline{x} \cdot \sum_{i=1}^{n} x_i \right)$$

$$= 4n \cdot \left(\sum_{i=1}^{n} x_i^2 - n \cdot \overline{x}^2 \right) \qquad \Big| \text{ see } (*)$$

$$= 4n \cdot \left(\sum_{i=1}^{n} (x_i - \overline{x})^2 \right) > 0$$

(*)

$$\sum_{i=1}^{n} (x_i - \overline{x})^2 = \sum_{i=1}^{n} (x_i^2 - 2x_i\overline{x} + \overline{x}^2) = \sum_{i=1}^{n} x_i^2 - 2\overline{x}\sum_{i=1}^{n} x_i + n\overline{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2\overline{x} \cdot n\overline{x} + n\overline{x}^2 = \sum_{i=1}^{n} x_i^2 - 2n\overline{x}^2 + n\overline{x}^2 = \sum_{i=1}^{n} x_i^2 - n\overline{x}^2$$