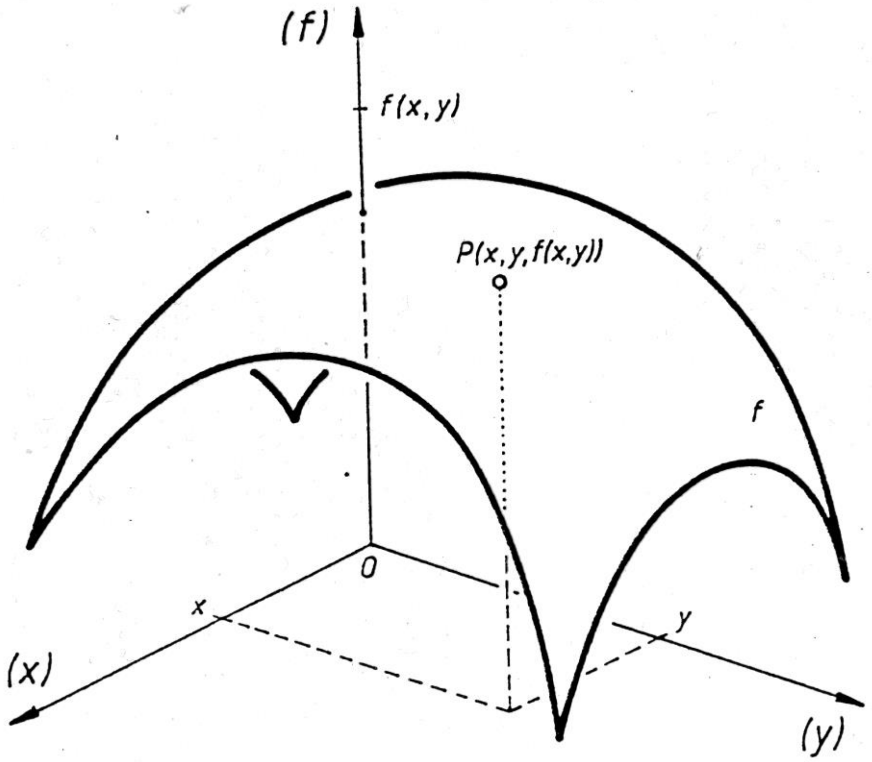
2.3 Multivariate functions

$$y = f(\underline{x}) = f(x_1 \dots , x_n) : Q \subset R^n \rightarrow R$$

Definite mapping of Q into R



2.3.1 Examples, Continuity

- Yield function
- Graph of the function is displayable as curved surface for n=2
- Paraboloid $y = x_1^2 + x_2^2$
- Objective function in the fodder-mix modell:
  minimal costs: $60x_1 + 45x_2 + 36x_3 = f(x_1, x_2, x_3)$
  is a linear function

Remarks concerning the continuity of functions

n = 1

f (x) continous in x_0: Is the distance of **x and x_o** small, the distance of **f (x) and f (x_0)** is small as well.

$$\Updownarrow$$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

n > 1 $\underline{x} = (x_1, x_2, ..., x_n) \in R^n$ $\underline{x}^0 = (x_1^0, x_2^0, ..., x_n^0) \in R^n$

f ($\underline{x}$) continous in $\underline{x}^0$: Is the distance of **$\underline{x}$ and $\underline{x}^0$** small, the distance of **f ($\underline{x}$) and f ($\underline{x}^0$)** is small as well.

$$\Updownarrow$$

$$\lim_{\underline{x} \to \underline{x}^0} f(\underline{x}) = f(\underline{x}^0)$$

Concerning this, the distance of n-tuples is defined using the Euclidean norm.

$$\| \underline{x} - \underline{x}^0 \| := \sqrt{\sum_{i=1}^{n} (x_i - x_i^0)^2} .$$

Heuristics: n = 1 Drawing of the curve without any break

n = 2 „Curved surface without any hole or crack“.

2.3.2 Partial derivative

Gradient of the function in direction of the axes

Partial derivative:

Derivative of the function $\mathbf{f(\underline{x}) = f(x_1, x_2, ..., x_n)}$ with respect **to one of those variables in each case**, whereby the other variables are concerned as constants.

Also the definition of the partial derivative is done at first at the local point $\underline{\mathbf{x}}^{\mathbf{0}} \in \mathbf{R}^{\mathbf{n}}$ by using a limit:

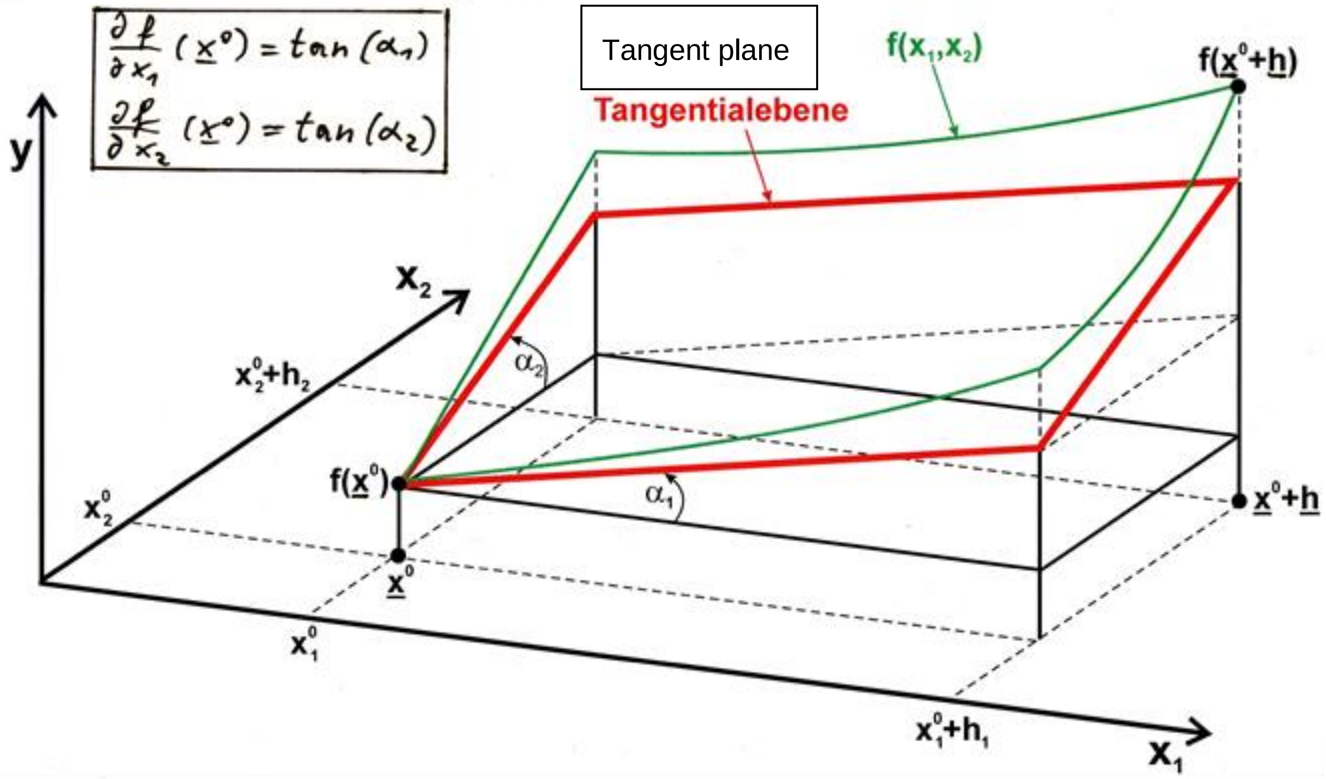
$$\lim_{h \to 0} \frac{f(x_1^0, ..., x_i^0 + h, ..., x_n^0) - f(x_1^0, ..., x_i^0, ..., x_n^0)}{h} \quad \text{for } i = 1, ..., n.$$

Designation: $\dfrac{\partial f}{\partial x_i}$ (also $\dfrac{\partial}{\partial x_i} f, \dfrac{\partial y}{\partial x_i}$ **respectively** f_{x_i}, f_i)

Geometrical interpretation:

Gradient of the function in direction of the axes.

Geometrische Interpretation der partiellen Ableitungen



2. Partial derivatives (Second order partial derivatives):

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$ as abbreviated form also f_{ij};

This means that the partial derivative of the function with respect to x_i, is partially derivated again with respect to the variable x_j.

Altogether there are n^2 second order derivative functions:

$$\begin{array}{cccc}
\frac{\partial^2 f}{\partial x_1^2}, & \frac{\partial^2 f}{\partial x_1 \, \partial x_2}, & \ldots, & \frac{\partial^2 f}{\partial x_1 \, \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \, \partial x_1}, & \frac{\partial^2 f}{\partial x_2^2}, & \ldots, & \frac{\partial^2 f}{\partial x_2 \, \partial x_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial^2 f}{\partial x_n \, \partial x_1}, & \frac{\partial^2 f}{\partial x_n \, \partial x_2}, & \ldots, & \frac{\partial^2 f}{\partial x_n^2}
\end{array}$$

Example 1: $\quad y = f(x_1, x_2) = x_1^2 + x_2^2$

$$\frac{\partial f}{\partial x_1} = 2x_1 \quad , \quad \frac{\partial f}{\partial x_2} = 2x_2$$

Example 2: $\quad f(x_1, x_2) = 5 + 2x_1 + 5x_1^2 + 8x_1 x_2 + 7x_2 + 5x_2^2$

$$\frac{\partial f}{\partial x_1} = 2 + 10x_1 + 8x_2 \quad , \quad \frac{\partial f}{\partial x_2} = 8x_1 + 7 + 10x_2 ,$$

$$\frac{\partial^2 f}{\partial x_1 \, \partial x_1} = 10 \qquad \frac{\partial^2 f}{\partial x_1 \, \partial x_2} = 8$$

$$\frac{\partial^2 f}{\partial x_2 \, \partial x_1} = 8 \qquad \frac{\partial^2 f}{\partial x_2 \, \partial x_2} = 10$$

2.3.3 Local extrema of multivariate functions

(1) One variable: $\mathbf{x \in R}$, $\mathbf{y = f(x)}$,

we are looking for a local maximum respectively minimum of the function f(x)

necessary condition: $\mathbf{f'(x) = 0}$,

sufficient condition: $\mathbf{f''(x) < 0}$ local maximum

bzw. $\mathbf{f''(x) > 0}$ local minimum.

(2) n variables: $\mathbf{(x_1, x_2, ..., x_n) = \underline{x} \in R^n}$

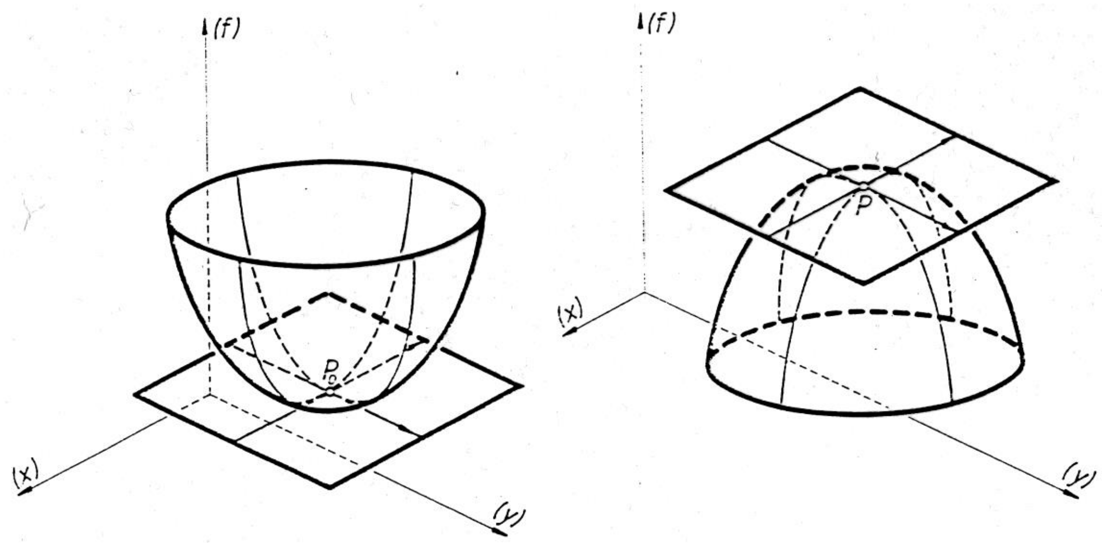
Definition:

The function f(x) has a local maximum(respectively minimum) at an inner point $\mathbf{\underline{x}^0}$ of the definition range if a real number $\delta > \mathbf{0}$ exists with $\mathbf{f(\underline{x}) \leq f(\underline{x}^0)}$ **(respectively** $\mathbf{f(\underline{x}) \geq f(\underline{x}^0)}$) for every $\mathbf{\underline{x} \in R^n}$ with $\| \mathbf{\underline{x} - \underline{x}^0} \| < \delta$.

Collective term: local extremum

$\left\{ \mathbf{\underline{x} \in R^n} \mid \| \mathbf{\underline{x} - \underline{x}^0} \| < \delta \right\}$ is called an open sphere around $\mathbf{\underline{x}^0 \in R^n}$

Local maximum/ Local Minimum



Necessary condition for relative extrema:

Theorem: If $\mathbf{f(\underline{x})}$ has a local extremum at point $\underline{\mathbf{x}}^{\mathbf{0}} \in \mathbf{R}^{\mathbf{n}}$ and if all partial derivatives $\frac{\partial \mathbf{f}}{\partial \mathbf{x_i}}$ exist at this point, than $\frac{\partial \mathbf{f}}{\partial \mathbf{x_i}} (\underline{\mathbf{x}}^{\mathbf{0}}) = \mathbf{0} \quad \forall \mathbf{i} = \mathbf{1, ..., n}$.

We consider n equations. The solutions of this system of equations are called stationary points.

Furthermore let n = 2, that means we have two variables $\mathbf{x_1, x_2}$.

So there is a geometrical interpretation of the necessary conditions:

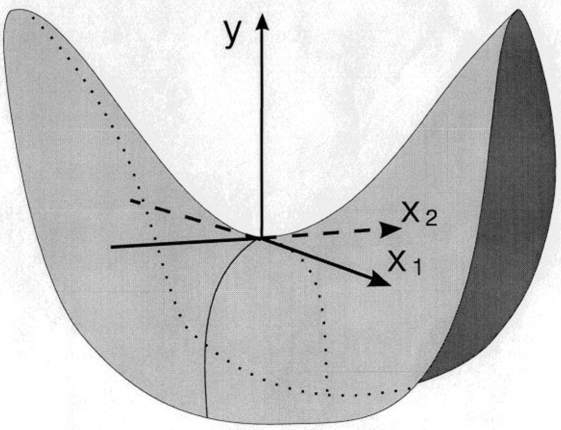
the tangent plane of $\mathbf{f(x_1, x_2)}$ at point $\underline{\mathbf{x}}^{\mathbf{0}} = (\mathbf{x_1^0, x_2^0})$ is parallel to the $\mathbf{x_1, x_2} -$plain.

In this case, we also need a sufficient condition for the local extremum.

Example: Saddle surface $\mathbf{f(x_1, x_2) = x_1 \cdot x_2}$

Sattelfläche $f(x_1, x_2) = x_1 \cdot x_2$

Saddle surface



Sufficient condition for a relative extremum:

Let $\mathbf{f(x_1,x_2)}$ be defined and continuous in a neighbourhood of a stationary point $\underline{\mathbf{x}}^{\mathbf{0}} = (\mathbf{x_1^0, x_2^0})$and let us suppose that all first and second partial derivatives exist and are continuous as well.

Theorem: $\mathbf{f(x_1,x_2)}$ has a local minimum at the stationary point

$\underline{\mathbf{x}}^{\mathbf{0}} = (\mathbf{x_1^0, x_2^0})$ if

(1) $$\mathbf{D} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x_1}\, \partial \mathbf{x_1}} (\underline{\mathbf{x}}^{\mathbf{0}}) \cdot \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x_2}\, \partial \mathbf{x_2}} (\underline{\mathbf{x}}^{\mathbf{0}}) - \left[\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x_1}\, \partial \mathbf{x_2}} (\underline{\mathbf{x}}^{\mathbf{0}}) \right]^2 > \mathbf{0}$$

and

(2) $$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x_1}\, \partial \mathbf{x_1}} (\underline{\mathbf{x}}^{\mathbf{0}}) > \mathbf{0}.$$

If D > 0 and $\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x_1}\, \partial \mathbf{x_1}} (\underline{\mathbf{x}}^{\mathbf{0}}) < \mathbf{0}$, then f has a local maximum at $\underline{\mathbf{x}}^{\mathbf{0}}$.

In Example 2 from section 2.3.2 we defined all first and second partial derivatives for the function

$$f(x_1, x_2) = 5 + 2x_1 + 5x_1^2 + 8x_1 x_2 + 7x_2 + 5x_2^2 \ .$$

The necessary condition is the linear equality system

$$2 + 10x_1 + 8x_2 = 0$$

$$7 + 8x_1 + 10x_2 = 0\,.$$

The solution is the stationary point $\underline{x}^0 = (1, -\frac{3}{2})$.

The sufficient condition is

(1) $D = 10 \cdot 10 - 8^2 = 36 > 0$ and

(2) $\dfrac{\partial^2 f}{\partial x_1 \, \partial x_1} = 10 > 0\,.$

(1) makes sure there is a local extremum at point $\underline{x}^0$,

(2) shows that the local extremum is a local minimum.