Fodder-mix-problem

(exercise after OHSE, D. (2004/2005): Mathematik für Wirtschaftswissenschaftler. Bd. I u. II, München: Vahlen)

Three different types of fodder (F_1, F_2, F_3) are designated to feed chicken on a farm.

These three types shall be mixed so that the feed contains

at least 80 units carbohydrates (C),

at least 120 units protein and (P)

at most 60 units fat (F).

The following table shows the price for one quantity unit of each fodder (F_1, F_2, F_3) and which amount of each nutrient one quantity unit contains.

	C	P	F	Price (€)
F_1	2	3	1	60
F_2	3	1	½	45
F_3	1	2	1	36

Which amount of each fodder (F_1, F_2, F_3) is required to get the lowest total cost?

The mathematical modelling of this exercise leads to this linear optimization problem.

$$\min \left\{ 60x_1 + 45x_2 + 36x_3 \;\middle|\; \begin{array}{l} 2x_1 + 3x_2 + x_3 \geq \; 80 \\ 3x_1 + x_2 + 2x_3 \geq 120, \; x_1, x_2, x_3 \geq 0 \\ x_1 + \frac{1}{2}x_2 + x_3 \leq \; 60 \end{array} \right\}$$

1. Set theoretic and arithmetic tools

1.1 Elements of set-theory

a) **Definition:**
A **set** is the bundling of objects. (To create a new object).

Symbolism:

1) Elements of a set: small latin letters

   A set: capital latin letters

2) $\mathbf{M} = \{\mathbf{x} \mid \mathbf{H(x)}\}$ The set is defined by the elements which comply with the condition or characteristic H (x)

3) $\mathbf{M} = \{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$ Enumeration of elements

4) $\mathbf{x_k} \in \mathbf{M}$ $\mathbf{x_k}$ is an element of M (k – Index).

5) $\exists$ Existence-Operator: It exists an element with...

   $\forall$ All-Operator: For all elements we assume...

b) Special sets

$\emptyset \quad \left(:= \{ \mathbf{x} \mid \mathbf{x} \neq \mathbf{x} \} \right)$	Empty set
$\mathbf{N} = \{ \mathbf{0, 1, 2, ...} \}$	Set of natural numbers
$\mathbf{Z} = \{ \mathbf{0}, +\mathbf{1}, -\mathbf{1}, +\mathbf{2}, -\mathbf{2}, \mathbf{...} \}$	Set of integer numbers
Q	Set of rational numbers (finite and recurring decimals)
R	Set of real numbers

Intervals play a key role as subsets of real numbers.

For real numbers a and b

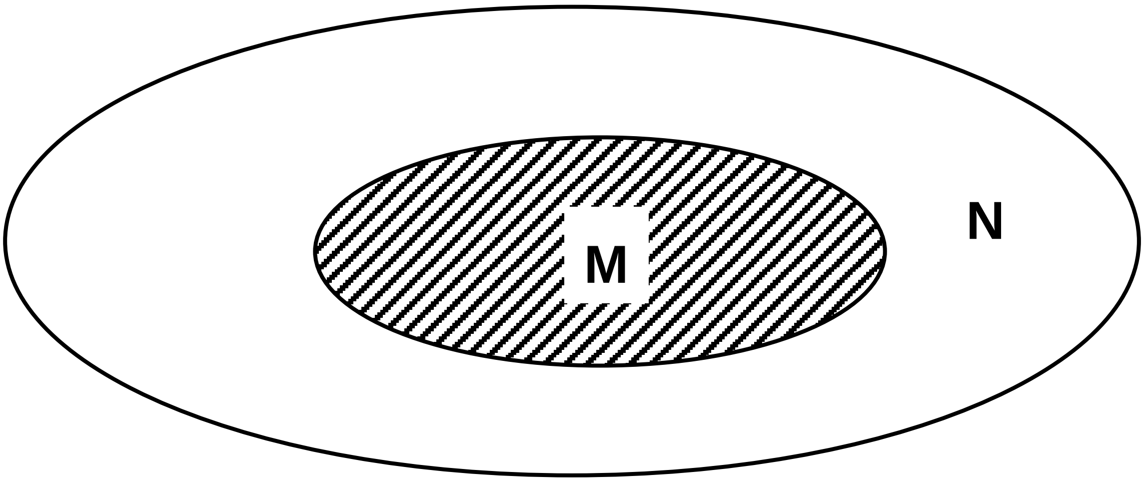
$\mathbf{(a,b)} \;:= \{ \mathbf{x} \in \mathbf{R} \mid \mathbf{a} < \mathbf{x} < \mathbf{b} \}$ describes an open interval,

$\mathbf{[a,b]} \;:= \{ \mathbf{x} \in \mathbf{R} \mid \mathbf{a} \leq \mathbf{x} \leq \mathbf{b} \}$ describes a closed interval.

c) Relations between sets

$\mathbf{M} \subset \mathbf{N}$ (M is **subset** of N) if and only if (abbreviation: iff.) each element of M is also an element of N.

VENN diagram:



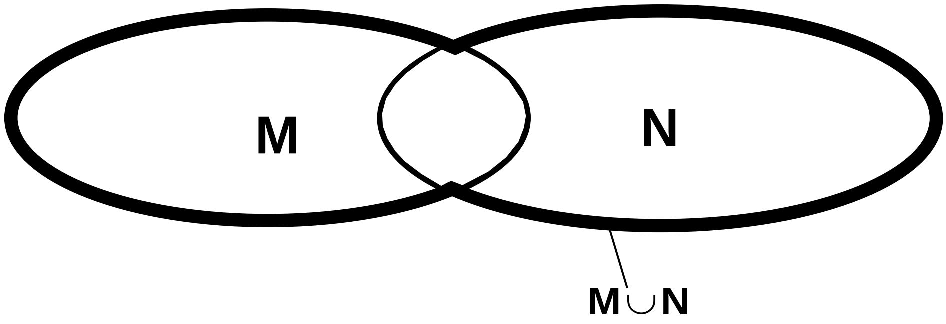
For each set M it is: $\emptyset \subset \mathbf{M}$.

$\mathbf{M} = \mathbf{N}$ iff. $\mathbf{M} \subset \mathbf{N}$ and $\mathbf{N} \subset \mathbf{M}$

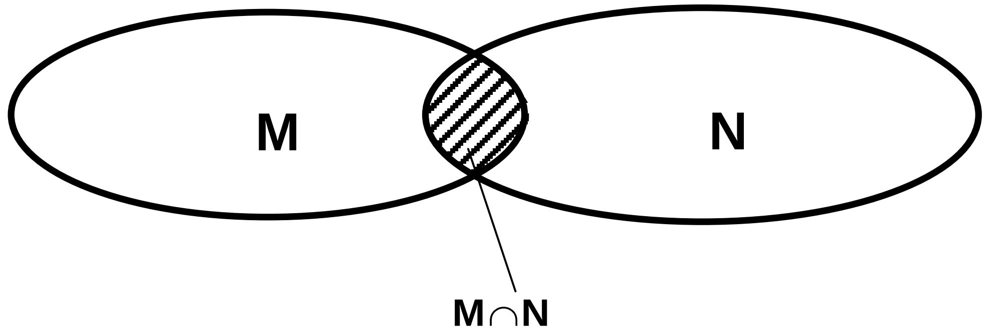
d) Operations with sets

(1) **Union** $M \cup N := \{ a \mid a \in M \ \underline{\textbf{OR}} \ a \in N \}$

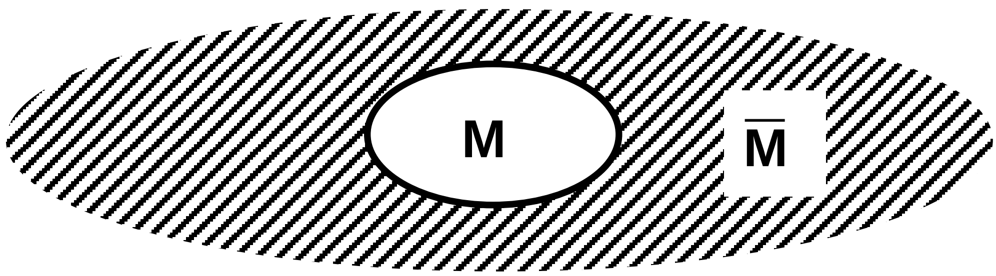
VENN diagram:



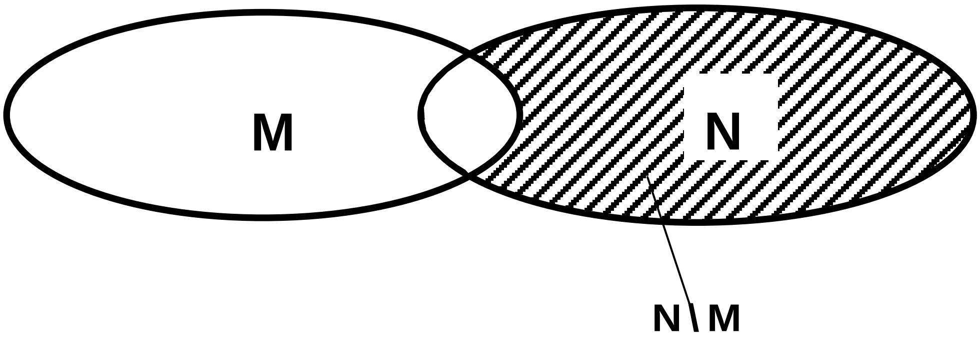
(2) **Intersection** $M \cap N := \{ a \mid a \in M \ \underline{\textbf{AND}} \ a \in N \}$



(3) **Complement** $\overline{M} := \{ a \mid a \notin M \}$



(4) **Set difference** $\mathbf{N \setminus M := \{a \mid a \in N \text{ and } a \notin M\}}$



(5) **Power set $\mathbf{2^M}$ of M** (Set of all sets of M)

$\mathbf{2^M := \{N \mid N \subset M\}}$

Example: $\mathbf{\{a,b\}}$

$$\mathbf{2^{\{a,b\}} = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}}$$

(6) **Ordered pair**

(a, b) ordered double-elemented pair

(a, b) = (c, d) iff. a = c and b = d

(f, g) = (3, -7) iff. f = 3 g = -7

(7) **Cartesian Product** (Cross product)

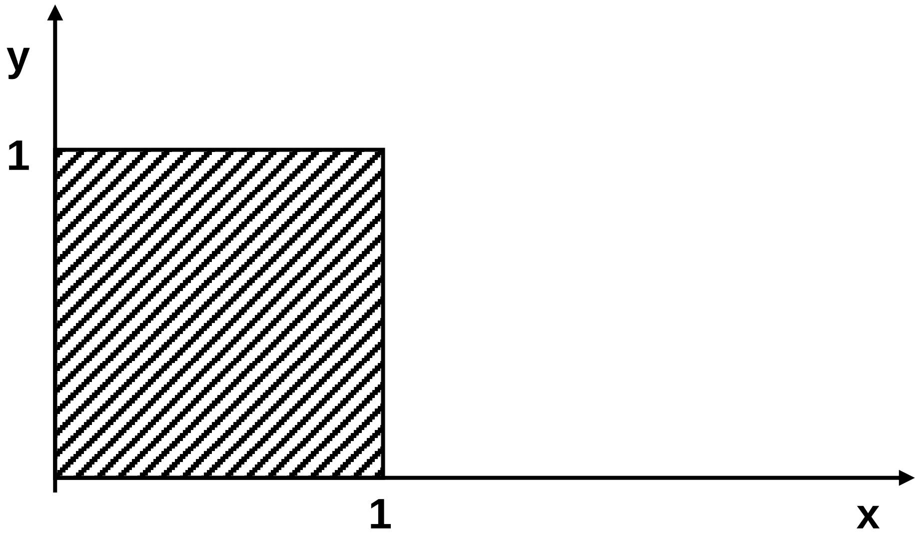
$\mathbf{M \times N := \{(a,b) \mid a \in M \text{ and } b \in N\}}$

Example:

1) $\mathbf{M = \{1,2\}, N = \{x,y\}, M \times N = \{(1,x), (1,y), (2,x), (2,y)\}}$

2) $\mathbf{X = \{x \mid 0 \le x \le 1\}}$

$\mathbf{Y = \{y \mid 0 \le y \le 1\}}$



3) $\mathbf{R \times R = R^2 := \{(x,y) \mid x \in R, y \in R\}}$

$\mathbf{R^3 := R \times R \times R}$

(8) f is called **mapping** of M into N iff. $\mathbf{f \subset M \times N}$.
Think about examples concerning the cross products mentioned above.

(9) **Inverse mapping** $\mathbf{f^{-1}}$ **of f** (also: Inverse of f)

$$\mathbf{f^{-1} := \{(a,b) \mid (b,\ a) \in f\}}$$

(10) **Function**

Definition:

A mapping f is called **function** or definite mapping of M in N iff.

there is for every $\mathbf{a \in M}$ exactly one element $\mathbf{b \in N}$

with $\mathbf{(a,b) \in f}$.

Another writing for $\mathbf{(a,b) \in f}$ is $\mathbf{b = f(a)}$

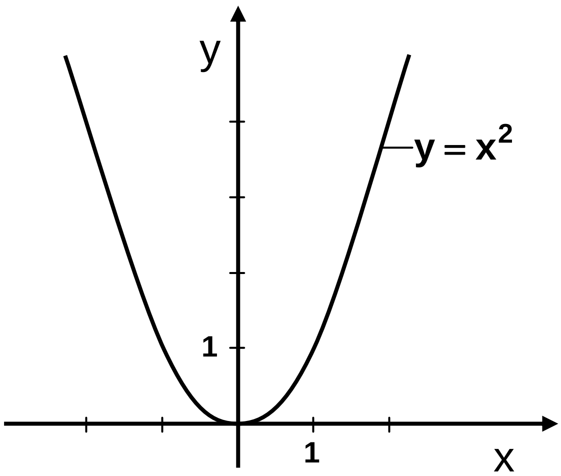
Domain: $\mathbf{M}$

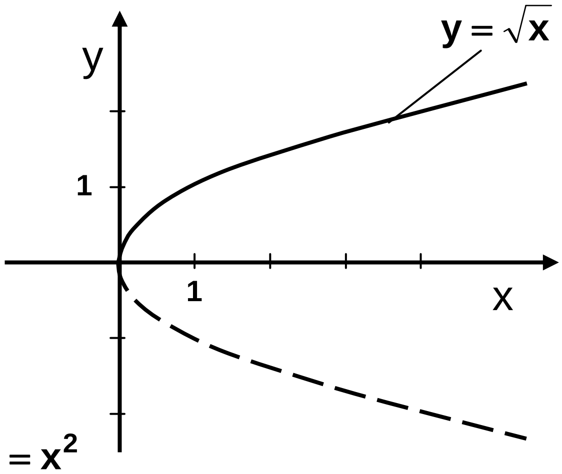
Range: $\mathbf{\{b \in N \mid \exists\ a \in M\ with\ (a,b) \in f\}}$

Different stages of generalization:

$y = x^2 + 3x - 2$	fixed parabola at $(-\frac{3}{2}, -\frac{17}{4})$
$y = x^2 + px + q$	parabola with the parameters p and q
$y = f(x)$	function

Example: $y = x^2$





The existing inverse mapping of $y = x^2$ is not a function.

Definition:

f is called **one-to-one mapping** iff. $\mathbf{f}$ **and** $\mathbf{f^{-1}}$ are functions.

Example: $\mathbf{y = x^2}$ **, for** $\mathbf{x \geq 0}$ ⟷ $\mathbf{y = \sqrt{x}}$ **, for** $\mathbf{x \geq 0}$

Functions which are inverse to each other
(inverse functions)